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Spin susceptibility of interacting two-dimensional electrons in the presence of spin-orbit coupling

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A long-range interaction via virtual particle-hole pairs between Fermi-liquid quasiparticles leads to a nonanalytic behavior of the spin susceptibility χ as a function of the temperature (T), magnetic field (\mathbf{B}), and wavenumber. In this paper, we study the effect of the Rashba spin-orbit interaction (SOI) on the nonanalytic behavior of χ for a two-dimensional electron liquid. Although the SOI breaks the $SU(2)$ symmetry, it does not eliminate nonanalyticity but rather makes it anisotropic: while the linear scaling of χ_{zz} with T and $|\mathbf{B}|$ saturates at the energy scale set by the SOI, that of χ_{xx} ($= \chi_{yy}$) continues through this energy scale, until renormalization of the electron-electron interaction in the Cooper channel becomes important. We show that the Renormalization Group flow in the Cooper channel has a non-trivial fixed point, and study the consequences of this fixed point for the nonanalytic behavior of χ . An immediate consequence of SOI-induced anisotropy in the nonanalytic behavior of χ is a possible instability of a second-order ferromagnetic quantum phase transition with respect to a first-order transition to an XY ferromagnetic state.

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I. INTRODUCTION

The issue of nonanalytic corrections to the Fermi liquid theory has been studied extensively in recent years.^{1,2} The interest to this subject is stimulated by a variety of topics, from intrinsic instabilities of ferromagnetic quantum phase transitions¹⁻⁵ to enhancement of the indirect exchange interaction between nuclear spins in semiconductor heterostructures with potential applications in quantum computing.⁶⁻⁸ The origin of the nonanalytic behavior can be traced to an effective long-range interaction of fermions via virtual particle-hole pairs with small energies and with momenta which are either small (compared to the Fermi momentum k_F) or near $2k_F$.^{9,10} In 2D, this interaction leads to a linear scaling of χ with a characteristic energy scale E , set by either the temperature T or the magnetic field $|\mathbf{B}|$, or else by the wavenumber of an inhomogeneous magnetic field $|\mathbf{q}|$ (whichever is larger when measured in appropriate units).¹¹⁻¹⁸ Higher-order scattering processes in the Cooper (particle-particle) channel result in additional logarithmic renormalization of the result: at the lowest energies, $\chi \propto E/\ln^2 E$.^{3,6,7,19-22} The sign of the effect, i.e., whether χ increases or decreases with E , turns out to be non-universal, at least in a generic Fermi liquid regime, i.e., away from the ferromagnetic instability: while the second-order perturbation theory predicts that χ increases with E , the sign of the effect may be reversed either due to a proximity to the Kohn-Luttinger superconducting instability^{6,7,21} or higher-order processes in the particle-hole channel.^{3,20}

In this paper, we explore the effect of the spin-orbit interaction (SOI) on the nonanalytic behavior of the spin susceptibility. The SOI is important for practically all systems of current interest at low enough energies; at

the same time, the nonanalytic behavior is also an inherently low-energy phenomenon. A natural question to ask is: what is the interplay between these two low-energy effects? At first sight, the SOI should regularize nonanalyticities at energy scales below the scale set by this coupling. (For a Rashba-type SOI, the relevant scale is given by the product $|\alpha|k_F$, where α is the coupling constant of the Rashba Hamiltonian; here and in the rest of the paper, we set \hbar and k_B to unity). Indeed, as we have already mentioned, the origin of the nonanalyticity is the long-range effective interaction originating from the singularities in the particle-hole polarization bubble. If, for instance, the temperature is the largest scale in the problem, these singularities are smeared by the temperature with an ensuing nonanalytic dependence of χ on T . On the other hand, if the Zeeman energy $\mu_B|\mathbf{B}|$ is larger than T , it provides a more efficient mechanism of regularization of the singularities and, as result, χ exhibits a nonanalytic dependence on $|\mathbf{B}|$. The same argument applies also to the $|\mathbf{q}|$ dependence. It is often said that the SOI plays the role of an effective magnetic field, which acts on electron spins. If so, one should expect, for instance, a duality between the T and $|\alpha|k_F$ scalings of χ (by analogy to a duality between T and $\mu_B|\mathbf{B}|$ scalings of χ), i.e., in a system with fixed SOI, the nonanalytic T dependence of χ should saturate at $T \sim |\alpha|k_F$. The main message of this paper is that such a duality does not, in fact, exist; more precisely, not all components of the susceptibility tensor exhibit the duality. In particular, the in-plane component of χ , χ_{xx} , continues to scale linearly with T and $\mu_B|\mathbf{B}|$, even if these energies are smaller than $|\alpha|k_F$. On the other hand, the T and $\mu_B|\mathbf{B}|$ dependences of χ_{zz} do saturate at $T \sim |\alpha|k_F$.

The reason for such a behavior is that although the SOI does play a role of the effective magnetic field, this field

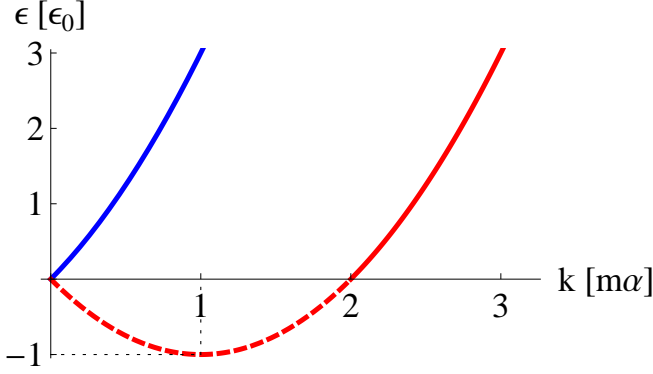


FIG. 1. (COLOR ONLINE) Rashba spectrum in zero magnetic field. The energy is measured in units of $\epsilon_0 \equiv m\alpha^2/2$, the momentum is measured in units of $m\alpha$.

depends on the electron momentum. To understand the importance of this fact, we consider the Rashba Hamiltonian in the presence of an external magnetic field,²³ which couples only to the spins of the electrons but not to their orbital degrees of freedom

$$\begin{aligned} H_R &= \frac{k^2}{2m} \hat{I} + \alpha(\boldsymbol{\sigma} \times \mathbf{k})_z + \frac{g\mu_B \boldsymbol{\sigma}}{2} \cdot \mathbf{B} \\ &= \frac{k^2}{2m} \hat{I} + \frac{g\mu_B \boldsymbol{\sigma}}{2} \cdot [\mathbf{B}_R(\mathbf{k}) + \mathbf{B}], \end{aligned} \quad (1.1)$$

where α is the SOI, \mathbf{k} is the electron momentum in the plane of a two-dimensional electron gas (2DEG), $\boldsymbol{\sigma}$ is a vector of Pauli matrices, \mathbf{e}_z is a normal to the plane, g is the gyromagnetic ratio, μ_B is the Bohr magneton, and the effective Rashba field, defined as $\mathbf{B}_R = (2\alpha/g\mu_B)(\mathbf{k} \times \mathbf{e}_z)$, is always in the 2DEG plane. The effective Zeeman energy is determined by the total magnetic field $\mathbf{B}_{\text{tot}} = \mathbf{B}_R + \mathbf{B}$ as

$$\bar{\Delta}_{\mathbf{k}} \equiv \frac{g\mu_B |\mathbf{B}_{\text{tot}}|}{2} = \sqrt{\alpha^2 k^2 + 2\alpha(\boldsymbol{\Delta} \times \mathbf{k})_z + \Delta^2}, \quad (1.2)$$

where we introduced $\boldsymbol{\Delta} = g\mu_B \mathbf{B}/2$ for the "Zeeman field", such that Δ is the Zeeman energy of an electron spin in the external magnetic field ($\hbar = 1$) and 2Δ equals to the Zeeman splitting between spin-up and spin-down states. A combined effect of the Rashba and external magnetic fields gives rise to two branches of the electron spectrum (see Fig. 1) with dispersions

$$\varepsilon_{\mathbf{k}}^{\pm} = \frac{k^2}{2m} \pm \bar{\Delta}_{\mathbf{k}}. \quad (1.3)$$

If $\mathbf{B} \parallel \mathbf{e}_z$, the external and effective magnetic fields are perpendicular to each other, as shown in Fig. 2a, so that the magnitude of the total magnetic field is $|\mathbf{B}_{\text{tot}}| = |\mathbf{B} + \mathbf{B}_R| = \sqrt{B^2 + B_R^2}$. This means that the external and magnetic fields are totally interchangeable, and the T dependence of the spin susceptibility is cut off by the largest of the two scales. However, if the external field is in the plane (and defines the x axis in Fig. 2b),

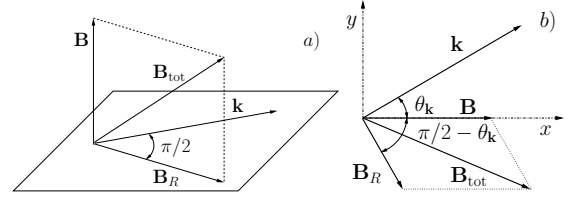


FIG. 2. Interplay between the external magnetic field \mathbf{B} and effective magnetic field \mathbf{B}_R due to the Rashba spin-orbit interaction. Left: the external field is perpendicular to the plane of motion. Right: the external field is in the plane of motion.

the magnitude of the total field depends on the angle $\theta_{\mathbf{k}}$ between \mathbf{k} and \mathbf{B} . In particular, for a weak external field,

$$\bar{\Delta}_{\mathbf{k}} \approx |\alpha|k + \Delta \sin \theta_{\mathbf{k}}. \quad (1.4)$$

If the electron-electron interaction is weak, the nonanalytic behavior of the spin susceptibility is due to particle-hole pairs with total momentum near $2k_F$, formed by electron and holes moving in almost opposite directions. In this case, the second term in Eq. (1.4) is of the opposite sign for electrons and holes. The effective Zeeman energy of the whole pair, formed by fermions from Rashba branches s and s' , is

$$\Delta_{\text{pair}} = s\bar{\Delta}_{\mathbf{k}} - s'\bar{\Delta}_{-\mathbf{k}} = (s - s')|\alpha|k + (s + s')\Delta \sin \theta_{\mathbf{k}}. \quad (1.5)$$

Only those pairs which "know" about the external magnetic field –via the second term in Eq. (1.5)– renormalize the spin susceptibility. According to Eq. (1.5), such pairs are formed by fermions from the same Rashba branch ($s = s'$). However, since the first term in Eq. (1.5) vanishes in this case, such pairs do not "know" about the SOI, which means that the Rashba and external magnetic fields are not interchangeable, and the SOI energy scale does not provide a cutoff for the T dependence of χ .

We now briefly summarize the main results of the paper. We limit our consideration to the 2D case and to the Rashba SO coupling, present in any 2D system with broken symmetry with respect to reversal of the normal to the plane. We focus on the dependencies of χ on T , α , and $|\mathbf{B}|$, deferring a detailed discussion of the dependence of χ on $|\mathbf{q}|$ to another occasion.²⁴ Throughout the paper, we assume that the SO coupling and electron-electron interaction, characterized by the coupling constant U , are weak, i.e., $|\alpha|k_F \ll E_F$ and $mU \ll 1$. The latter condition implies that only $2k_F$ scattering processes are relevant for the nonanalytic behavior of the spin susceptibility. Accordingly, U is the $2k_F$ Fourier transform of the interaction potential. Renormalization provides another small energy scale at which the product $(mU/2\pi) \ln(\Lambda/T_C)$ (where Λ is an ultraviolet cutoff of the theory) becomes comparable to unity, i.e., $T_C \equiv \Lambda \exp(-2\pi/mU)$. Depending on the ratio of the two small energy scales $|\alpha|k_F$ and T_C –different behaviors are possible.

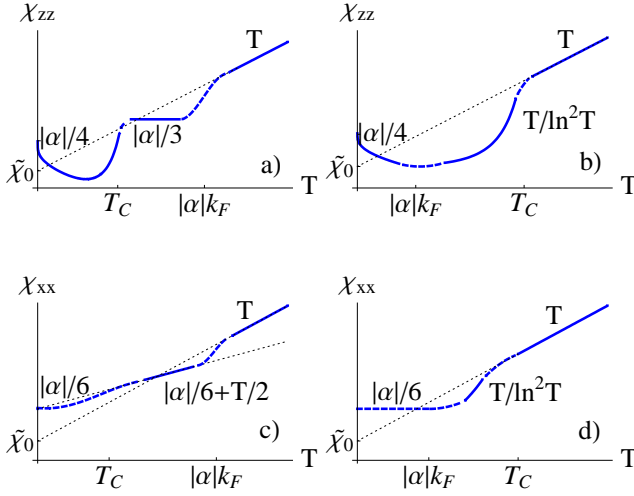


FIG. 3. (COLOR ONLINE) A sketch of the temperature dependence of the spin susceptibility for the transverse (top) and in-plane (bottom) magnetic field. Dashed segments in crossovers between various asymptotic regimes do not represent results of actual calculations. The left (right) panel is valid for $T_C \ll |\alpha|k_F$ ($T_C \gg |\alpha|k_F$) where $T_C \equiv \Lambda \exp(-2\pi/mU)$ is the temperature below which renormalization in the Cooper channel becomes significant. The zero temperature limit for interacting electrons, denoted by $\tilde{\chi}_0$, is given by Eq. (2.8) in the Random Phase Approximation.

In Fig. 3a, we sketch the T dependence of χ_{zz} for the case of $T_C \ll |\alpha|k_F$. For $T \gg |\alpha|k_F$, χ_{zz} scales linearly with T , in agreement with previous studies; a correction due to the SOI is on the order of $(\alpha k_F/T)^2$ [cf. Eq. (3.29)]. For $T_C \ll T \ll |\alpha|k_F$, χ_{zz} saturates at a value proportional to $|\alpha|$; the correction due to finite T is on the order of $(T/|\alpha|k_F)^3$ [cf. Eq. (3.30)]. For $T \lesssim T_C$, renormalization in the Cooper channel becomes important. In the absence of the SOI, the coupling constant of the electron-electron interaction in the Cooper channel flows to zero as $U/|\ln T|$. Consequently, the T scaling of the spin susceptibility changes to $T/\ln^2 T$ for $T \ll T_C$. In the presence of the SOI, the situation is different. We show that the Renormalization Group (RG) flow of U in this case has a non-trivial fixed point characterized by finite value of the electron-electron coupling, which is only numerically smaller than its bare value. In between these two limits, the coupling constant changes non-monotonically with $\ln T$, and so does χ_{zz} . In both high- and low T limits (compared to T_C), however, χ_{zz} is almost T independent, so Cooper renormalization affects the $|\alpha|$ term in χ_{zz} .

The T dependence of χ_{zz} for $T_C \gg |\alpha|k_F$ is sketched in Fig. 3b. In this case, the crossover between the T and $T/\ln^2 T$ forms occurs first, at $T \sim T_C$, while the $T/\ln^2 T$ form crosses over to $|\alpha|$ at $T \sim |\alpha|k_F$.

We now turn to χ_{xx} . For $T_C \ll |\alpha|k_F$, its T dependence is shown in Fig. 3c. The high- T behavior is again linear [also with a $(\alpha k_F/T)^2$ correction, cf. Eq. (3.50)]. For $T_C \ll T \ll |\alpha|k_F$, this behavior changes to χ_{xx}

$\propto |\alpha|k_F/6 + T/2$, which means that χ_{xx} continues to decrease with T with a slope half of that at higher T [cf. Eq. (3.51)]. Finally, for $T \lesssim T_C$, Cooper renormalization leads to the same $|\alpha|$ dependence as for χ_{zz} but with a different prefactor. The behavior of χ_{xx} for $|\alpha|k_F \ll T_C$ is shown in Fig. 3d. Apart from the numbers, this behavior is similar to that of χ_{zz} in that case.

The dependences of χ_{xx} and χ_{zz} on the external magnetic field can be obtained (up to a numerical coefficient) simply by replacing T by Δ in all formulas presented above. In particular, if Cooper renormalization can be ignored, both χ_{xx} and χ_{zz} scale linearly with Δ for $\Delta \gg |\alpha|k_F$ but only χ_{xx} continues to scale with Δ for $\Delta \ll |\alpha|k_F$. A linear scaling of χ_{ii} with Δ implies the presence of a nonanalytic, $|M_i|^3$ term in the free energy, where \mathbf{M} is the magnetization and $i = x, y, z$. Consequently, while the cubic term is isotropic ($F \propto |\mathbf{M}|^3$) for larger M (so that the corresponding Zeeman energy is above $|\alpha|k_F$), it is anisotropic at smaller $|M|$: $F \propto |M_x|^3 + |M_y|^3$. A negative cubic term in F implies metamagnetism and an instability of the second-order ferromagnetic quantum phase transition toward a first-order one.^{1,3} An anisotropic cubic term implies anisotropic metamagnetism, i.e., a phase transition in a finite magnetic field, if it is applied along the plane of motion, but no transition for a perpendicular field, and also that the first-order transition is into an XY rather than Heisenberg ferromagnetic state. This issue is discussed in more detail in Sec. V.

The rest of the paper is organized as follows. In Sec. II, we formulate the problem and discuss the T dependence of the spin susceptibility for free electrons with Rashba spectrum (more details on this subject are given in Appendix A). Section III A explains the general strategy of extracting the nonanalytic behavior of χ from the thermodynamic potential in the presence of the SOI. The second-order perturbation theory for the temperature and magnetic-field dependences of the transverse and in-plane susceptibilities is presented in Secs. III B and III C, respectively. In Sec. III D, we show that, as is also the case in the absence of the SOI, there is no contribution to the nonanalytic behavior of the spin susceptibility from processes with small momentum transfers, including the transfers commensurate with (small) Rashba splitting of the free-electron spectrum (more details on this issue are provided in Appendix B). Renormalization of spin susceptibility in the Cooper channel is considered in Sec. IV. An explicit calculation of the third-order Cooper contribution to χ_{zz} is shown in Sec. IV B. In Sec. IV C 2 we derive the RG flow equations for the scattering amplitudes in the absence of the magnetic field; the effect of the finite field on the RG flow is discussed in Appendix C. The effect of Cooper-channel renormalization on the nonanalytic behavior of χ_{zz} and χ_{xx} is discussed in Secs. IV C 3 and IV C 4, correspondingly. Implications of our results in the context of quantum phase transitions are discussed in Ref. V, where we also give our conclusions.

II. SPIN SUSCEPTIBILITY OF FREE RASHBA FERMIONS

In this Section, we set the notations and discuss briefly the properties of Rashba electrons in the absence of the electron-electron interaction. The Hamiltonian describing a two-dimensional electron gas (2DEG) in the presence of a Rashba SOI and an external magnetic field \mathbf{B} is given by Eq. (1.1).

In the following, we consider two orientations of the magnetic field: transverse ($\mathbf{B} = B\mathbf{e}_z$) and parallel ($\mathbf{B} = B\mathbf{e}_x$) to the 2DEG plane. It is important to emphasize that, when discussing the perpendicular magnetic field, we neglect its orbital effect. Certainly, if the spin susceptibility is measured as a response to an external magnetic field, its orbital and spin effects cannot be separated. However, there are situations when the spin part of χ_{zz} is of primary importance. For example, the Ruderman-Kittel-Kasuya-Yosida (RKKY) interaction between the local moments located in the 2DEG plane arises only from the spin susceptibility of itinerant electrons, because the orbital effect of the dipolar magnetic field of such moments is negligible. In this case, χ_{xx} and χ_{zz} determine the strength of the RKKY interaction between two moments aligned along the x and z axis, respectively. Also, divergences of χ_{zz} and χ_{xx} signal ferromagnetic transitions into states with easy-axis and easy-plane anisotropies, respectively. Since it is this kind of physical situations we are primarily interested in this paper, we will ignore the orbital effect of the field from now on.

The Green's function corresponding to the Hamiltonian (1.1) is obtained by matrix inversion

$$\hat{G}_K \equiv \frac{1}{i\omega - \hat{H}} = \sum_{s=\pm} \hat{\Omega}_s(\mathbf{k}) g_s(K), \quad (2.1)$$

where we use the “relativistic” notation $K \equiv (\omega, \mathbf{k})$ with ω being a fermionic Matsubara frequency, the matrix $\hat{\Omega}_s(\mathbf{k})$ is defined as

$$\hat{\Omega}_s(\mathbf{k}) \equiv \frac{1}{2} \left(\hat{I} + s\hat{\zeta} \right), \quad (2.2a)$$

$$\hat{\zeta} = \frac{\alpha(k_y\hat{\sigma}_x - k_x\hat{\sigma}_y) + \hat{\sigma} \cdot \boldsymbol{\Delta}}{\Delta_{\mathbf{k}}}, \quad (2.2b)$$

$g_s(K) = 1/(i\omega - \xi_{\mathbf{k}} - s\bar{\Delta}_{\mathbf{k}})$ is the single-electron Green's function, $\xi_{\mathbf{k}} \equiv \frac{k^2}{2m} - \mu$, μ is the chemical potential, and $\bar{\Delta}_{\mathbf{k}}$ is given by Eq. (1.2).

As we have already pointed out in the Introduction, an important difference between the cases of transverse and in-plane magnetic field is the dependence of the effective Zeeman energy [Eq. (1.2)] on the electron momentum. For the transverse magnetic field, ($\Delta_x = \Delta_y = 0, \Delta_z = \Delta$), the Zeeman energy is isotropic in the momentum space and quadratic in Δ in the weak-field limit: $\bar{\Delta}_{\mathbf{k}} \approx |\alpha|k_F + \Delta^2/2|\alpha|k_F$. Correspondingly, the Fermi surfaces of Rashba branches are concentric circles with slightly (in proportion to Δ^2) different radii. For the in-plane magnetic field, ($\Delta_x = \Delta, \Delta_y = \Delta_z = 0$), the effective

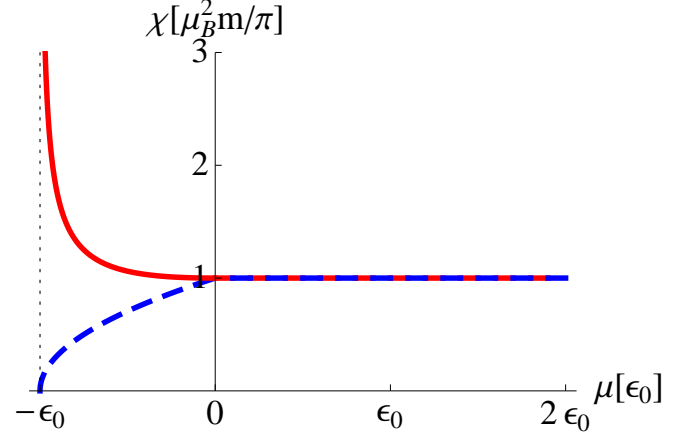


FIG. 4. (COLOR ONLINE) Spin susceptibility of free Rashba fermions (in units of $\chi_0 = \mu_B^2 m / \pi$) as a function of the chemical potential μ (in units of $\epsilon_0 = m\alpha^2/2$). Solid (red): $\chi_{xx}^{(0)}$; dashed (blue): $\chi_{zz}^{(0)}$. Note that $\chi_{xx}^{(0)} \neq \chi_{zz}^{(0)}$ if only the lowest Rashba branch is occupied ($-\epsilon_0 \leq \mu < 0$) but $\chi_{xx}^{(0)} = \chi_{zz}^{(0)} = \chi_0$ if both branches are occupied ($\mu > 0$) at $T = 0$.

Zeeman energy is anisotropic in the momentum space, cf. Eq. (1.4). Correspondingly, the Fermi surfaces of Rashba branches are also anisotropic and their centers are shifted by finite momentum, proportional to Δ_x .

We now give a brief summary of results for the susceptibility in the absence of electron-electron interaction, which sets the zeroth order of the perturbation theory (for more details, see Appendix A). The in-plane rotational symmetry of the Rashba Hamiltonian guarantees that $\chi_{yy}^{(0)} = \chi_{xx}^{(0)}$. The static uniform susceptibility (defined in the limit of zero frequency and vanishingly small wavenumber) is still diagonal even in the presence of the SOI: $\chi_{ij}^{(0)} = \delta_{ij}\chi_{ii}^{(0)}$, although $\chi_{xx}^{(0)} = \chi_{yy}^{(0)} \neq \chi_{zz}^{(0)}$ in general. The susceptibility depends strongly on whether both or only the lower of the two Rashba branches are occupied, see Fig. (4). In the latter case, the spin response is strongly anisotropic. At $T = 0$,

$$\chi_{zz}^{(0)} = \chi_0 \sqrt{1 + \mu/\epsilon_0} \quad (2.3a)$$

$$\chi_{xx}^{(0)} = \chi_0 \frac{1 + \mu/2\epsilon_0}{\sqrt{1 + \mu/\epsilon_0}}, \quad (2.3b)$$

where $\chi_0 \equiv \mu_B^2 m / \pi$ is the spin susceptibility of 2D electrons in the absence of the SOI, $\epsilon_0 = m\alpha^2/2$ is the depth of the energy minimum of the lower branch and the chemical potential μ is within the range $-\epsilon_0 \leq \mu \leq 0$. The in-plane susceptibility exhibits a 1D-like van Hove singularity at the bottom of the lower branch, i.e., for $\mu \rightarrow -\epsilon_0$. On the other hand, if both branches are occupied (which is the case for $\mu > 0$), the spin susceptibility is isotropic and the same as in the absence of the SOI

$$\chi_{zz}^{(0)} = \chi_{xx}^{(0)} = \chi_0. \quad (2.4)$$

This isotropy can be related to a hidden symmetry of

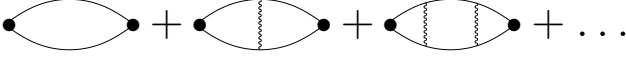


FIG. 5. The RPA diagrams for the spin susceptibility corresponding to Eq. (2.7).

the Rashba Hamiltonian manifested by conservation of the square of the electron's velocity operator \hat{v} .²⁵ The eigenvalue of \hat{v}^2 , given by $2\epsilon/m + 2\alpha^2$ with ϵ being the energy, is the same for both branches. The square of the group velocity $v_g^2 = (\nabla_{\mathbf{k}}\epsilon_{\mathbf{k}}^{\pm})^2 = 2\epsilon/m + \alpha^2$ also does not depend on the branch index. Therefore, the total density of states

$$\nu(\epsilon) = \frac{1}{2\pi} \frac{k^+ + k^-}{|v_g|} = \frac{m}{\pi}, \quad (2.5)$$

where $k^{\pm} = \mp m\alpha + \sqrt{m^2\alpha^2 + 2m\epsilon}$ are the momenta of

the \pm branches corresponding to energy ϵ , is the same as without the SOI, if both branches are occupied. One can show also that isotropy of the spin susceptibility is not specific for the Rashba coupling but is there also in the presence of both Rashba and (linear) Dresselhaus interactions.²⁶

As one step beyond the free-electron model, we consider a Stoner-like enhancement of the spin susceptibility by the electron-electron interaction. In the absence of the SOI and for a point-like interaction U , the renormalized spin susceptibility is given by

$$\chi = \frac{\chi_0}{1 - mU/\pi}. \quad (2.6)$$

In the presence of the SOI, the ladder series for the susceptibility, shown in Fig. 5 is given by

$$\begin{aligned} \chi_{ii} = \chi_{ii}^{\{0\}} + U\mu_B^2 \sum_{K,P} \text{Tr} \left[\hat{\sigma}_i \hat{G}(K+Q) \hat{G}(P+Q) \hat{\sigma}_i \hat{G}(P) \hat{G}(K) \right] \\ - U^2 \mu_B^2 \sum_{K,P,L} \text{Tr} \left[\hat{\sigma}_i \hat{G}(K+Q) \hat{G}(P+Q) \hat{G}(L+Q) \hat{\sigma}_i \hat{G}(L) \hat{G}(P) \hat{G}(P) \right] + \dots \end{aligned} \quad (2.7)$$

where $Q = (\Omega = 0, \mathbf{q} \rightarrow 0)$ and $i = x, z$. Although the traces do look different for χ_{xx} and χ_{zz} , these differences disappear after angular integrations, and the resulting series are the same. As we have shown above, the zero-order susceptibilities are also the same (and equal to χ_0) if both Rashba branches are occupied; hence

$$\chi_{ii} = \mu_B^2 \left(\frac{m}{\pi} + \frac{m^2 U}{\pi^2} + \frac{m^3 U^2}{\pi^3} + \dots \right) = \frac{\chi_0}{1 - mU/\pi}, \quad (2.8)$$

which is the same result as in Eq. (2.6). Therefore, at the mean-field level, the spin susceptibility remains isotropic and independent of the SOI. In the rest of the paper, we will show that none of these two features survives beyond the mean-field level: the actual spin susceptibility is anisotropic and both its components do depend on the SOI.

We now come back to the free-electron model and discuss the T dependence of the spin susceptibility. A special feature of a 2DEG in the absence of the SOI is a breakdown of the Sommerfeld expansion at finite T : since the density of states does not depend on the energy, all power-law terms of this expansion vanish, and the resulting T dependence of $\chi^{\{0\}}$ is only exponential. The SOI leads to the energy dependence of the density of states for the individual branches, and one would expect the Sommerfeld expansion to be restored. This is what indeed happens if only the lower branch is occupied. In

this case,

$$\chi_{zz}^{\{0\}}(T) = \chi_{zz}^{\{0\}}(0) - \chi_0 \frac{\pi^2}{24} \left(\frac{T}{\epsilon_0} \right)^2 \frac{1}{(1 + \mu/\epsilon_0)^{3/2}} \quad (2.9a)$$

$$\chi_{xx}^{\{0\}}(T) = \chi_{xx}^{\{0\}}(0) + \chi_0 \frac{\pi^2}{48} \left(\frac{T}{\epsilon_0} \right)^2 \frac{2 - \mu/\epsilon_0}{(1 + \mu/\epsilon_0)^{5/2}}, \quad (2.9b)$$

provided that $T \ll \min\{-\mu, \epsilon_0 + \mu\}$. [Here, $\chi_{zz}^{\{0\}}(0)$ and $\chi_{xx}^{\{0\}}(0)$ are the zero temperature values given by Eqs. (2.3a) and (2.3b)]. However, if both branches are occupied, the energy dependent terms in the branch densities of states cancel out, and the resulting dependence is exponential, similar to the case of no SOI, although with different pre-exponential factors:

$$\chi_{zz}^{\{0\}}(T) = \chi_0 \left(1 - \frac{T}{2\epsilon_0} e^{-\mu/T} \right) \quad (2.10a)$$

$$\chi_{xx}^{\{0\}}(T) = \chi_0 \left(1 + \frac{T^2}{4\epsilon_0^2} e^{-\mu/T} \right) \quad (2.10b)$$

for $T \ll \epsilon_0 \ll \mu$, and

$$\chi_{zz}^{\{0\}}(T) = \chi_0 \left[1 - \left(1 - \frac{2\epsilon_0}{3T} \right) e^{-\mu/T} \right] \quad (2.11a)$$

$$\chi_{xx}^{\{0\}}(T) = \chi_0 \left[1 - \left(1 - \frac{4\epsilon_0}{3T} \right) e^{-\mu/T} \right] \quad (2.11b)$$

for $\epsilon_0 \ll T \ll \mu$. In a similar way, one can show that that there are no power-law terms in the dependence of $\chi_{xx}^{\{0\}}$ and $\chi_{zz}^{\{0\}}$ on the external magnetic field.

Notice that $\chi_{xx} \neq \chi_{zz}$ at finite temperature, even if both Rashba subbands are occupied. This suggests that the hidden symmetry of the Rashba Hamiltonian is, in fact, a rotational symmetry in a $(2+1)$ space with imaginary time being an extra dimension.²⁷ Finite temperature should then play a role of finite size along the time axis breaking the rotational symmetry.

In what follows, we assume that the SOI is weak, i.e., $|\alpha| \ll v_F$, and thus the energy scales describing SOI are small: $m\alpha^2 \ll |\alpha|k_F \ll \mu$. This condition also means that both Rashba branches are occupied. [Also, from now on we relabel $\mu \rightarrow E_F$.] The main result of this section is that, for a weak SO coupling, the T and Δ dependences of χ in the free case are at least exponentially weak and thus cannot mask the power-law dependences arising from the electron-electron interaction, which are discussed in the rest of the paper.

III. SPIN SUSCEPTIBILITY TO SECOND ORDER IN THE ELECTRON-ELECTRON INTERACTION

A. General strategy

The spin susceptibility tensor χ_{ij} is related to the thermodynamic (grand canonical) potential $\Xi(T, \alpha, \Delta)$ by the following identity

$$\chi_{ij}(T, \alpha) = - \frac{\partial^2 \Xi}{\partial B_i \partial B_j} \Big|_{B=0}. \quad (3.1)$$

Equation (3.5) describes the interaction among electrons from all Rashba branches via an effective vertex $U(|\mathbf{k} - \mathbf{p}|)B_{ss'}(\mathbf{k}, \mathbf{p})$, which depends not only on the momentum transfer $\mathbf{k} - \mathbf{p}$ but also on the initial momenta \mathbf{k} and \mathbf{p} themselves. This last dependence is due to anisotropy of the Rashba spinors.

It has been shown in Refs. 3 and 15 that, at weak coupling, the main contribution to the nonanalytic part of the spin susceptibility comes from “backscattering” processes, i.e., processes with $\mathbf{p} \approx -\mathbf{k}$ and small q (compared to k_F). In particular, the second-order contribution is entirely of the backscattering type. The proof given in Ref. 3 applies to any kind of the angular-dependent vertex and, thus, also to vertices in Eq. (3.5). Therefore, the calculation can be simplified dramatically by putting in $\mathbf{p} = -\mathbf{k}$ and neglecting \mathbf{q} in the effective vertices. The last assumption is justified as long as the typical values of

To second order in the electron-electron interaction $U(q)$, there is only one diagram for the thermodynamic potential that gives rise to a nonanalytic behavior: diagram *a*) in Fig. 6. The rest of the diagrams in this figure can be shown to be irrelevant (cf. Sec. IIID). Algebraically, diagram *a*) in Fig. 6 reads

$$\delta\Xi^{(2)} \equiv -\frac{1}{4} \sum_Q \sum_K \sum_P U^2(|\mathbf{k} - \mathbf{p}|) \times \text{Tr}(\hat{G}_K \hat{G}_P) \text{Tr}(\hat{G}_{K+Q} \hat{G}_{P+Q}), \quad (3.2)$$

where $\sum_Q = T \sum_\Omega \int d^2q / (2\pi)^2$, $\sum_K = T \sum_\omega \int d^2k / (2\pi)^2$, and we use “relativistic” notation $K \equiv (\omega, \mathbf{k})$ with a fermionic frequency ω and $Q \equiv (\Omega, \mathbf{q})$ with a bosonic frequency Ω .

Evaluation of the first spin trace in Eq. (3.2) yields

$$\text{Tr}(\hat{G}_K \hat{G}_P) = \frac{1}{2} \sum_{ss'} B_{ss'}(\mathbf{k}, \mathbf{p}) g_s(K) g_{s'}(P), \quad (3.3)$$

where

$$B_{ss'}(\mathbf{k}, \mathbf{p}) \equiv 1 + ss' \frac{\alpha^2 \mathbf{k} \cdot \mathbf{p} + \alpha [\boldsymbol{\Delta} \times (\mathbf{k} + \mathbf{p})]_z + \Delta^2}{\bar{\Delta}_{\mathbf{k}} \bar{\Delta}_{\mathbf{p}}} \quad (3.4)$$

and $\bar{\Delta}_{\mathbf{k}}$ is given by Eq. (1.2). The second-order thermodynamic potential then becomes

$$\delta\Xi^{(2)} \equiv -\frac{1}{16} \sum_Q \sum_K \sum_P U^2(|\mathbf{k} - \mathbf{p}|) B_{s_1 s_3}(\mathbf{k}, \mathbf{p}) B_{s_2 s_4}(\mathbf{k} + \mathbf{q}, \mathbf{p} + \mathbf{q}) g_{s_1}(K) g_{s_2}(P) g_{s_3}(K + Q) g_{s_4}(P + Q). \quad (3.5)$$

q are determined by the smallest energy of the problem, i.e., $q \sim \max\{T/v_F, m|\alpha|\} \ll k_F$. By the same argument, the magnitudes of \mathbf{k} and \mathbf{p} in the vertices can be replaced by k_F . The bare interaction is then evaluated at $|\mathbf{k} - \mathbf{p}| \approx 2k_F$, and we introduce a coupling constant $U \equiv U(2k_F)$. Ignoring the angular dependence of $B_{s_i s_j}$ for a moment, Eq. (3.5) is reduced to a convolution of two particle-hole bubbles, formed by electrons belonging to either the same or different Rashba branches

$$\Pi_{s_i s_j}(Q) = \sum_K g_{s_i}(K) g_{s_j}(K + Q). \quad (3.6)$$

By assumption, both components of Q in $\Pi_{s_i s_j}(Q)$, i.e., Ω and q , are small (compared to E_F and k_F , correspondingly). It is important to realize that, despite a two-band nature of the Rashba spectrum, $\Pi_{s_i s_j}(Q)$ has no threshold-like singularities at the momentum $q_0 = 2m\alpha$,

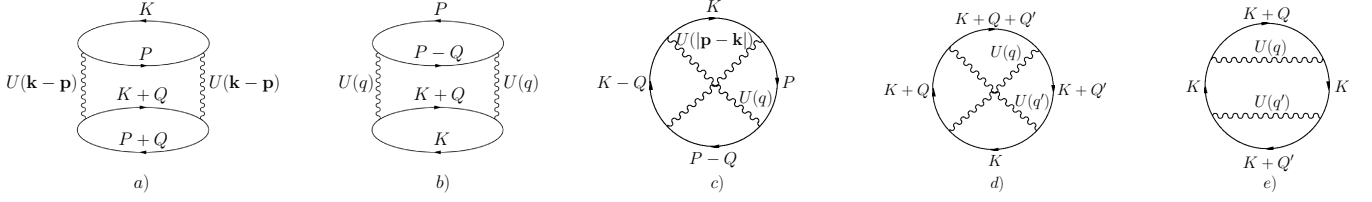


FIG. 6. Second order diagrams for the thermodynamic potential. A nonanalytic contribution comes only from diagram *a*), where the momenta are arranged in such a way that $\mathbf{k} \approx -\mathbf{p}$ while Q is small; therefore, $k \approx p \approx k_F$, and the momentum transfer in each scattering event is close to $|\mathbf{k} - \mathbf{p}| \approx 2k_F$. As is shown in Sec. III D, diagrams *b*)-*e*) do not contribute to the nonanalytic behavior of the spin susceptibility.

separating the Rashba subbands²⁸ (for a detailed derivation of this result, see Appendix B). Therefore, the non-analytic behavior of the spin susceptibility comes only from the Landau-damping singularity of the dynamic

bubble, as it is also the case in the absence of the SOI.

After the simplifications described above, Eq. (3.5) becomes

$$\delta\Xi^{(2)} \equiv -\frac{U^2}{16} \sum_Q \sum_K \sum_P B_{s_1 s_3} B_{s_2 s_4} g_{s_1}(K) g_{s_2}(P) g_{s_3}(K+Q) g_{s_4}(P+Q), \quad (3.7)$$

where

$$B_{ss'} \equiv B_{ss'}(\mathbf{k}_F, -\mathbf{k}_F) = 1 + ss' \frac{\Delta^2 - \alpha^2 k_F^2}{\Delta_{\mathbf{k}_F} \bar{\Delta}_{-\mathbf{k}_F}} \quad (3.8)$$

and $\mathbf{k}_F \equiv (\mathbf{k}/k)k_F$. Finally, to obtain the leading T dependence of χ_{ij} , it suffices to replace the fermionic Matsubara sums in Eq. (3.7) by integrals but keep the bosonic Matsubara sum as it is. The rest of the calculations proceed somewhat differently for the cases of the transverse and in-plane magnetic fields.

B. Transverse magnetic field

First, we consider a simpler case of the magnetic field transverse to the 2DEG plane: $\mathbf{\Delta} = g\mu_B B \mathbf{e}_z/2$. In this case, the effective Zeeman energy is isotropic in the momentum space; therefore, $\bar{\Delta}_{\mathbf{k}_F} = \bar{\Delta}_{-\mathbf{k}_F}$ and

$$B_{ss'} = 1 + ss' \frac{\Delta^2 - \alpha^2 k_F^2}{\bar{\Delta}_{\mathbf{k}_F}^2} = 1 + ss' \frac{\Delta^2 - \alpha^2 k_F^2}{\Delta^2 + \alpha^2 k_F^2}. \quad (3.9)$$

Thereby the integrals over d^3K and d^3P separate, and one obtains

$$\delta\Xi_{zz}^{(2)} = -\frac{U^2}{16} \sum_{s_1 \dots s_4 = \pm 1} B_{s_1 s_3} B_{s_2 s_4} \sum_Q \Pi_{s_1 s_2}(Q) \Pi_{s_3 s_4}(Q), \quad (3.10)$$

where $\Pi_{s_i s_j}(Q)$ is given by Eq. (3.6). The single-particle spectrum in the second Green's function in Eq. (3.6) can be linearized with respect to q as $\xi_{\mathbf{k}+\mathbf{q}} \approx \xi_{\mathbf{k}} + v_F q \cos \phi_{kq}$ with $\phi_{\mathbf{kq}} \equiv \angle(\mathbf{k}, \mathbf{q})$. Since, by assumption, the SOI is

small, the effective Zeeman energy in the Green's functions can be replaced by its value at $k = k_F$. Integration over dkk can be then replaced by that over $d\xi_{\mathbf{k}}$. The integrals over ω , $\xi_{\mathbf{k}}$ and ϕ_{kq} are performed in the same way as in the absence of the SOI, and we arrive at the following expression for the dynamic part of the polarization bubble

$$\Pi_{ss'} = \frac{m}{2\pi} \frac{|\Omega|}{\sqrt{[\Omega + i(s' - s)\bar{\Delta}_{\mathbf{k}_F}]^2 + (v_F q)^2}}. \quad (3.11)$$

We pause here for a comment. The polarization bubble in Eq. (3.11) is very similar to the dynamic part of the polarization bubble for spin-up and -down electrons in the presence of the magnetic field but in the absence of the SOI³

$$\Pi_{\uparrow\downarrow}(\Omega, q) = \frac{m}{2\pi} \frac{|\Omega|}{\sqrt{(\Omega - ig\mu_B B)^2 + (v_F q)^2}}. \quad (3.12)$$

As we already mentioned in Sec. I, the nonanalytic behavior of the spin susceptibility is due to an effective interaction of Fermi-liquid quasiparticles via particle-hole pairs with small energies and momenta near $2k_F$. Our calculation is arranged in such a way that, on a technical level, we deal with pairs with small momenta q . The spectral weight of these pairs is proportional to the polarization bubble, which is singular for small Ω and q . Since finite magnetic field cuts off the singularity in $\Pi_{\uparrow\downarrow}(\Omega, q)$, the nonanalytic dependence of χ on, e.g., temperature, saturates when T becomes comparable to the Zeeman splitting $g\mu_B B$. At lower energies, χ exhibits a nonanalytic dependence on Δ : $\chi \propto |\Delta|$. Likewise, the singularity in Eq. (3.11) is cut at the effective Zeeman energy $\bar{\Delta}_{\mathbf{k}_F}$, which reduces to the SOI energy scale $|\alpha|k_F$,

when the real magnetic field goes to zero. Therefore, one should expect the nonanalytic T dependence of χ_{zz} to be cut by the SOI. Although soft particle-hole pairs can be still generated within a given branch, i.e., for $s = s'$, the entire dependence on the Zeeman energy in this case is eliminated and processes of this type do not affect the spin susceptibility. In the rest of this section, we are going to demonstrate that the SOI indeed plays a role of the magnetic field for χ_{zz} .

For later convenience, we define a new quantity

$$\mathcal{P}_s \equiv \mathcal{P}_s(\Omega, q) = \frac{1}{\sqrt{(\Omega - 2is\bar{\Delta}_{\mathbf{k}_F})^2 + v_F^2 q^2}} \quad (3.13)$$

and sum over the Rashba branches in Eq. (3.10). The contribution of the set $\{s_1 = s_2 \equiv s, s_3 = s_4 \equiv s'\}$ does not depend on $\bar{\Delta}_{\mathbf{k}_F}$:

$$\sum_{s,s'} B_{ss'}^2 \Pi_{ss}^2 = \left(\frac{m\Omega}{2\pi}\right)^2 \sum_{s,s'} B_{ss'}^2 \mathcal{P}_0^2. \quad (3.14)$$

The set $\{s_1 = -s_2 \equiv s, s_3 = -s_4 \equiv s'\}$ gives

$$\sum_{s,s'} B_{ss'} B_{-s,-s'} \Pi_{s,-s} \Pi_{s',-s'} = \left(\frac{m\Omega}{2\pi}\right)^2 \sum_{s,s'} B_{ss'}^2 \mathcal{P}_s \mathcal{P}_{s'}, \quad (3.15)$$

where we used that $B_{ss} = B_{-s,-s}$. Finally, the sets $\{s_1 = s_2 \equiv s, s_3 = -s_4 \equiv s'\}$ and $\{s_1 = -s_2 \equiv s, s_3 = s_4 \equiv s'\}$ contribute

$$\begin{aligned} & \sum_{s,s'} (B_{ss'} B_{s,-s'} \Pi_{ss} \Pi_{s',-s'} + B_{ss'} B_{-s,s'} \Pi_{s,-s} \Pi_{s's'}) \\ &= 2 \left(\frac{m\Omega}{2\pi}\right)^2 \sum_{ss'} B_{ss'} B_{s,-s'} \mathcal{P}_0 \mathcal{P}_{s'}, \end{aligned} \quad (3.16)$$

where we relabeled the indices in the second sum ($s \rightarrow s'$, $s' \rightarrow s$) and used the symmetry property $B_{ss'} = B_{s's}$.

The angular integral contributes a unity, so that

$$\begin{aligned} \delta\Xi_{zz}^{(2)} = & - \left(\frac{mU}{8\pi}\right)^2 T \sum_{\Omega} \Omega^2 \int \frac{dq q}{2\pi} \\ & \times \sum_{ss'} [B_{ss'}^2 (\mathcal{P}_0^2 + \mathcal{P}_s \mathcal{P}_{s'}) + 2B_{ss'} B_{s,-s'} \mathcal{P}_0 \mathcal{P}_{s'}]. \end{aligned} \quad (3.17)$$

Now we sum over ss' , add and subtract a combination $2(\alpha^4 k_F^4 + 4\alpha^2 k_F^2 \Delta^2 + \Delta^4) \mathcal{P}_0^2$ inside the square brackets,

and obtain, after some algebra,

$$\begin{aligned} \delta\Xi_{zz}^{(2)} = & - \left(\frac{mU}{8\pi}\right)^2 T \sum_{\Omega} \Omega^2 \int_0^\infty \frac{dq q}{2\pi} \frac{4}{\bar{\Delta}_{\mathbf{k}_F}^4} [4\bar{\Delta}_{\mathbf{k}_F}^4 \mathcal{P}_0^2 \\ & + \Delta^4 (\mathcal{P}_+^2 + \mathcal{P}_-^2 - 2\mathcal{P}_0^2) + 4\alpha^2 k_F^2 \Delta^2 \mathcal{P}_0 (\mathcal{P}_+ + \mathcal{P}_- - 2\mathcal{P}_0)], \end{aligned} \quad (3.18)$$

where we used that $\int dq q (\mathcal{P}_+ \mathcal{P}_- - \mathcal{P}_0^2) = 0$. The first term in the square brackets in Eq. (3.18) does not depend on the effective field $\bar{\Delta}_{\mathbf{k}_F}$ and, therefore, can be dropped. Integration over q in the remaining terms is performed as

$$\int dq q \mathcal{P}_0 (\mathcal{P}_+ + \mathcal{P}_- - 2\mathcal{P}_0) = \frac{1}{v_F^2} \ln \frac{\Omega^2}{\Omega^2 + \bar{\Delta}_{\mathbf{k}_F}^2}, \quad (3.19)$$

$$\int dq q (\mathcal{P}_+^2 + \mathcal{P}_-^2 - 2\mathcal{P}_0^2) = \frac{1}{v_F^2} \ln \frac{\Omega^2}{\Omega^2 + 4\bar{\Delta}_{\mathbf{k}_F}^2}. \quad (3.20)$$

Collecting all terms together, we obtain

$$\begin{aligned} \delta\Xi_{zz}^{(2)} = & - \frac{2}{\pi} \left(\frac{mU}{4\pi v_F}\right)^2 \left[\frac{\Delta^4}{4\bar{\Delta}_{\mathbf{k}_F}^4} T \sum_{\Omega} \Omega^2 \ln \frac{\Omega^2}{\Omega^2 + 4\bar{\Delta}_{\mathbf{k}_F}^2} \right. \\ & \left. + \frac{\alpha^2 k_F^2 \Delta^2}{\bar{\Delta}_{\mathbf{k}_F}^4} T \sum_{\Omega} \Omega^2 \ln \frac{\Omega^2}{\Omega^2 + \bar{\Delta}_{\mathbf{k}_F}^2} \right]. \end{aligned} \quad (3.21)$$

The bosonic sum is evaluated by replacing the sum by an integral as follows

$$T \sum_{\Omega} F(\Omega) = \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \coth \frac{\Omega}{2T} \text{Im} \left[\lim_{\delta \rightarrow 0} F(-i\Omega + \delta) \right] \quad (3.22)$$

and using the identity

$$\begin{aligned} \text{Im} \left[\lim_{\delta \rightarrow 0} (-i\Omega + \delta)^2 \ln \frac{(-i\Omega + \delta)^2}{(-i\Omega + \delta)^2 + x^2} \right] \\ = \pi \Omega^2 \text{sign} \Omega \Theta(x^2 - \Omega^2), \end{aligned} \quad (3.23)$$

where $\Theta(x)$ stands for the step function.

The thermodynamic potential then becomes

$$\begin{aligned} \delta\Xi_{zz}^{(2)} = & - \frac{2}{\pi} \left(\frac{mU}{4\pi v_F}\right)^2 T^3 \left[\frac{\Delta^4}{4\bar{\Delta}_{\mathbf{k}_F}^4} \mathcal{F} \left(\frac{2\bar{\Delta}_{\mathbf{k}_F}}{T} \right) \right. \\ & \left. + \frac{\alpha^2 k_F^2 \Delta^2}{\bar{\Delta}_{\mathbf{k}_F}^4} \mathcal{F} \left(\frac{\bar{\Delta}_{\mathbf{k}_F}}{T} \right) \right] \end{aligned} \quad (3.24)$$

with

$$\mathcal{F}(y) = \int_0^y dx x^2 \coth(x/2) = -(1/3)y^2 [y + 6\text{Li}_1(e^y)] + 4[\zeta(3) + y\text{Li}_2(e^y) - \text{Li}_3(e^y)], \quad (3.25)$$

where $\text{Li}_n(z) \equiv \sum_{k=1}^{\infty} z^k / k^n$ is the polylogarithm func-

tion and $\zeta(z)$ is the Riemann zeta function. In practice,

the integral form of $\mathcal{F}(y)$ is more convenient as it can be easily expanded in the limits of small and large argument. Indeed, for $y \ll 1$, one expands $\coth(x/2)$ as $\coth(x/2) = 2/x + x/6$ and, upon integrating over x in

Eq. (3.25), obtains

$$\mathcal{F}(y) = y^2 + \frac{y^3}{24} + \mathcal{O}(y^4), \quad \text{for } 0 < y \ll 1. \quad (3.26)$$

For $y \gg 1$, one subtracts unity from the integrand and replaces the upper limit in the remaining integral by infinity:

$$\begin{aligned} \mathcal{F}(y) &= \frac{y^3}{3} + \int_0^y dx x^2 \left(\coth \frac{x}{2} - 1 \right) = \frac{y^3}{3} + \int_0^\infty dx x^2 \left(\coth \frac{x}{2} - 1 \right) - \int_y^\infty dx x^2 \left(\coth \frac{x}{2} - 1 \right) \\ &= \frac{y^3}{3} + 4\zeta(3) + \mathcal{O}(e^{-y}), \quad \text{for } y \gg 1. \end{aligned} \quad (3.27)$$

1. Temperature dependence of χ_{zz}

The (linear) spin susceptibility is given by $\chi_{zz} = -\partial^2 \Xi / \partial B_z^2|_{B=0}$, which means that only terms proportional to Δ^2 in the thermodynamic potential matter. Therefore, for finite α , the spin susceptibility comes entirely from the second term in the square brackets of Eq. (3.24), which is proportional to $\alpha^2 k_F^2 \Delta^2$. [On the other hand, for $\alpha = 0$ the second term vanishes, while $\Delta^4 / \bar{\Delta}_{\mathbf{k}_F}^4 = 1$, and the spin susceptibility comes exclusively from the first term: $\delta \Xi_{zz}^{(2)}$ still depends on the magnetic field through $\mathcal{F}(\bar{\Delta}_{\mathbf{k}_F}/T)$, where the Δ -dependence must be retained; in this case, $\mathcal{F}(\bar{\Delta}_{\mathbf{k}_F}/T)$ is evaluated as shown in Ref. 15.] Neglecting the first term and differentiating the second one, we obtain the interaction correction to χ_{zz} for $\Delta \rightarrow 0$ as

$$\delta \chi_{zz}^{(2)} = 2\chi_0 \left(\frac{mU}{4\pi} \right)^2 \frac{T^3}{\alpha^2 k_F^2 E_F} \mathcal{F} \left(\frac{|\alpha| k_F}{T} \right). \quad (3.28)$$

For $T \gg |\alpha| k_F$, the asymptotic expansion of \mathcal{F} in Eq. (3.26) gives

$$\delta \chi_{zz}^{(2)} \approx 2\chi_0 \left(\frac{mU}{4\pi} \right)^2 \left[\frac{T}{E_F} + \frac{1}{24} \frac{\alpha^2 k_F^2}{T E_F} + \dots \right]. \quad (3.29)$$

The first term in Eq. (3.29) coincides with the result of Refs. 3, 15, 21, and 22 obtained in the absence of the SOI, while the second term is a correction due to the finite SOI. In the opposite limit, i.e., for $T \ll |\alpha| k_F$, the asymptotic expansion of \mathcal{F} in Eq. (3.27) gives

$$\delta \chi_{zz}^{(2)} \approx 2\chi_0 \left(\frac{mU}{4\pi} \right)^2 \left[\frac{|\alpha| k_F}{3E_F} + 4\zeta(3) \frac{T^3}{\alpha^2 k_F^2 E_F} + \dots \right]. \quad (3.30)$$

As it was anticipated, the SOI cuts off the nonanalytic T dependence for $T \lesssim |\alpha| k_F$. However, the T dependence is replaced by a nonanalytic $|\alpha|$ dependence on the SO coupling.

Normalizing Eq. (3.28) to the leading T dependent term for $\alpha = 0$, i.e., by $\delta \chi_{zz}^{(2)}(T, \alpha = 0) =$

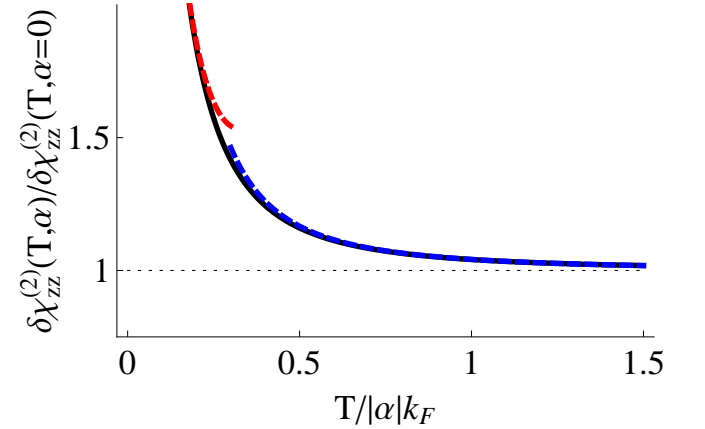


FIG. 7. (COLOR ONLINE) The second-order nonanalytic correction to χ_{zz} , normalized to its value in the absence of the SOI, as a function of $T/|\alpha|k_F$ [cf. Eq. (3.31)]. The asymptotic forms, given by Eqs. (3.29) and (3.29), are shown by dashed (red and blue) lines.

$\chi_0 (mU/4\pi)^2 (T/E_F)$, we express $\delta \chi_{zz}^{(2)}$ via a scaling function of the variable $T/|\alpha|k_F$

$$\frac{\delta \chi_{zz}^{(2)}(T, \alpha)}{\delta \chi_{zz}^{(2)}(T, \alpha = 0)} = \left(\frac{T}{|\alpha| k_F} \right)^2 \mathcal{F} \left(\frac{|\alpha| k_F}{T} \right). \quad (3.31)$$

The left-hand side of Eq. (3.31) is plotted in Fig. 7 along with its high and low T asymptotic forms.

Now we can give a physical interpretation of the above results. Although the electron-electron interaction mixes Rashba branches, the final result in Eq. (3.28) comes only from a special combination of electron states. Namely, three out of four electron states involved in the scattering process (two for the incoming and two for the outgoing electrons) must belong to the same Rashba branch, while the last one must belong to the opposite branch, as shown in Fig. 8a. This can be seen from Eq. (3.18) by considering four terms in the square brackets. The first term, proportional to $\bar{\Delta}_{\mathbf{k}_F}^4$, does not depend on the field upon the

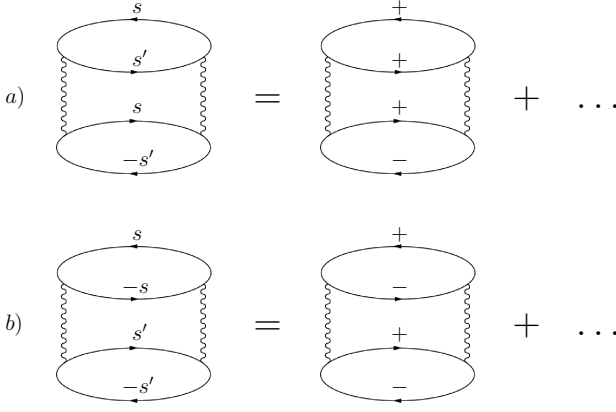


FIG. 8. Top: Diagrams contributing to nonanalytic behavior of χ_{zz} . There are eight such diagrams with the following choice of Rashba indices: $+++-, +-+-, +--+,-++-$, $--+-, --+-, -+--$, and $+---$; one of them is shown on the right. Bottom: Diagrams contributing to nonanalytic behavior of χ_{xx} . There are four such diagrams with the following choice of Rashba indices: $+-+-, +-+-, -++-$, and $-++-$; one of them is shown on the right.

cancellation with an overall factor of $\bar{\Delta}_{\mathbf{k}_F}^4$ in the denominator; the second term, proportional to $\alpha^4 k_F^4$, vanishes; the third term is already proportional to Δ^4 and, thus, cannot affect the spin susceptibility for finite α . Therefore, the effect comes exclusively from the last term, proportional to Δ^2 , because the product $B_{ss'}B_{s,-s'}$ is equal to $4\Delta^2(\alpha k_F)^2/(\alpha^2 k_F^2 + \Delta^2)^2 \approx 4\Delta^2/(\alpha k_F)^2$ for any choice of s and s' . This term corresponds to the structure $\sum_{ss'} B_{ss'}B_{s,-s'}\Pi_{s,-s}\Pi_{s's'}$ in Eq. (3.17). The diagrams corresponding to this structure are shown in Fig. 8a. Pairing electron Green's functions from different bubbles, we always obtain a combination $\sum_K g_{\pm}(K)g_{\pm}(K+Q)$, which depends neither on the SOI nor on the magnetic field, and a combination $\sum_K g_{\pm}(K)g_{\mp}(K+Q)$, which depends on both via the effective Zeeman energy $\bar{\Delta}_{\mathbf{k}_F} = \sqrt{\alpha^2 k_F^2 + \Delta^2}$. In the weak-field limit, one needs to keep Δ^2 only in the prefactor. The singularity in the combination $\sum_K g_{\pm}(K)g_{\mp}(K+Q)$ is then regularized by finite SOI, which is the reason why the nonanalytic T dependence is cut off by the SOI.

2. Magnetic-field dependence of χ_{zz}

Now we consider the case of $T \ll \max\{\Delta, |\alpha|k_F\}$ when, to first approximation, one can set $T = 0$. In this case, one can define a non-linear susceptibility $\chi_{zz}(B_z, \alpha) = -\partial^2 \Xi_{zz} / \partial B_z^2$ evaluated at finite rather than zero magnetic field. For $T \rightarrow 0$, we replace the scaling function \mathcal{F} in Eq. (3.24) by the first term in its large-argument asymptotic form (3.27) to obtain

$$\delta \Xi_{zz}^{(2)} = -\frac{2}{3\pi} \left(\frac{mU}{4\pi v_F} \right)^2 \frac{2\Delta^4 + \alpha^2 k_F^2 \Delta^2}{\sqrt{\Delta^2 + \alpha^2 k_F^2}}. \quad (3.32)$$

Differentiating twice with respect to the field, we find

$$\delta \tilde{\chi}_{zz}^{(2)}(B_z, \alpha) = \chi_0 \left(\frac{mU}{4\pi} \right)^2 \frac{|\Delta|}{E_F} \mathcal{G} \left(\frac{|\alpha|k_F}{\Delta} \right), \quad (3.33)$$

where

$$\mathcal{G}(x) = \frac{2x^6 + 23x^4 + 30x^2 + 12}{3(1+x^2)^{5/2}} \quad (3.34)$$

has the following asymptotics

$$\begin{aligned} \mathcal{G}(x \ll 1) &= 4 + x^4/6 + \dots \\ \mathcal{G}(x \gg 1) &= (2/3)|x| + 6/|x| + \dots \end{aligned} \quad (3.35)$$

For $|\Delta| \gg |\alpha|k_F$, the nonanalytic correction $\tilde{\chi}_{zz}(B_z, \alpha)$ reduces to the result of Ref. 3, obtained in the absence of the SOI, plus a correction term

$$\delta \chi_{zz}^{(2)}(B_z, \alpha) = 4\chi_0 \left(\frac{mU}{4\pi} \right)^2 \left[\frac{|\Delta|}{E_F} + \frac{1}{24} \frac{\alpha^4 k_F^4}{|\Delta|^3 E_F} + \dots \right]. \quad (3.36)$$

In the opposite limit of $|\Delta| \ll |\alpha|k_F$, the nonanalytic field dependence is cut off by the SOI

$$\delta \tilde{\chi}_{zz}^{(2)}(B_z, \alpha) = \frac{2}{3} \chi_0 \left(\frac{mU}{4\pi} \right)^2 \left[\frac{|\alpha|k_F}{E_F} + 9 \frac{\Delta^2}{|\alpha|k_F E_F} + \dots \right]. \quad (3.37)$$

C. In-plane magnetic field

If the magnetic field is along the x -axis, $\mathbf{\Delta} = \mu_B B \mathbf{e}_x / 2$, the effective Zeeman energies, $\bar{\Delta}_{\pm \mathbf{k}_F} \equiv \sqrt{\alpha^2 k_F^2 \pm 2\alpha k_F \sin \theta_{\mathbf{k}} \Delta + \Delta^2}$, depends on the angle $\theta_{\mathbf{k}}$ between \mathbf{k} and the direction of the field, chosen as the x axis. Coming back to Eq. (3.7), we integrate first over the fermionic frequencies, then over the magnitudes of the fermionic momenta, then over the angle between \mathbf{p} and \mathbf{q} , and finally over the angle between \mathbf{q} and \mathbf{k} (at fixed \mathbf{k}). This yields

$$\delta \Xi_{xx}^{(2)} = -\frac{U^2}{16} T \sum_{\Omega} \int \frac{d\theta_{\mathbf{k}}}{2\pi} \int \frac{dq}{2\pi} \times \sum_{\{s_i\}} B_{s_1 s_3} B_{s_2 s_4} \Pi_{s_1 s_2}^{+\mathbf{k}_F} \Pi_{s_3 s_4}^{-\mathbf{k}_F}, \quad (3.38)$$

where

$$\begin{aligned} \Pi_{ss'}^{\pm \mathbf{k}_F} &\equiv \Pi_{ss'}^{\pm \mathbf{k}_F}(\Omega, q; \theta_{\mathbf{k}}) = \sum_K' g_s^{\pm \mathbf{k}_F}(K) g_{s'}^{\pm \mathbf{k}_F}(K+Q) \\ &= \frac{m}{2\pi} \frac{|\Omega|}{\sqrt{[\Omega + i(s'-s)\bar{\Delta}_{\pm \mathbf{k}_F}]^2 + v_F^2 q^2}}, \end{aligned} \quad (3.39)$$

with $g_s^{\pm \mathbf{k}_F}(K) = 1/(i\omega - \xi_{\mathbf{k}} - s\bar{\Delta}_{\pm \mathbf{k}_F})$, and \sum_K' indicates that the integration over $\theta_{\mathbf{k}}$ is excluded. The remaining integration over $\theta_{\mathbf{k}}$ is performed last, after integration over q and summation over Ω .

As in Sec. III B, it is convenient to define a new quantity

$$\mathcal{P}_s^{\pm \mathbf{k}_F} \equiv \mathcal{P}_s^{\pm \mathbf{k}_F}(\Omega, q; \theta_{\mathbf{k}}) = \frac{1}{\sqrt{(\Omega - 2is\bar{\Delta}_{\pm \mathbf{k}_F})^2 + v_F^2 q^2}}, \quad (3.40)$$

and to re-write the thermodynamic potential as

$$\delta \Xi_{xx}^{(2)} = - \left(\frac{mU}{8\pi} \right)^2 T \sum_{\Omega} \Omega^2 \int \frac{d\theta_{\mathbf{k}}}{2\pi} \int \frac{dq}{2\pi} \sum_{ss'} \left[B_{ss'} B_{s,-s'} \mathcal{P}_0 (\mathcal{P}_{s'}^{+\mathbf{k}_F} + \mathcal{P}_{s'}^{-\mathbf{k}_F}) + B_{ss'}^2 (\mathcal{P}_0^2 + \mathcal{P}_s^{+\mathbf{k}_F} \mathcal{P}_{s'}^{-\mathbf{k}_F}) \right]. \quad (3.41)$$

Subsequently, we sum over s and s' , add and subtract $4(\bar{\Delta}_{\mathbf{k}_F}^2 \bar{\Delta}_{-\mathbf{k}_F}^2 - 2\alpha^2 k_F^2 \sin^2 \theta_{\mathbf{k}}) \mathcal{P}_0^2$ inside the square brackets, and, after some algebraic manipulations, obtain

$$\begin{aligned} \delta \Xi_{xx}^{(2)} = & -2 \left(\frac{mU}{8\pi} \right)^2 T \sum_{\Omega} \Omega^2 \int \frac{d\theta_{\mathbf{k}}}{2\pi} \int \frac{dq}{2\pi} \left[8\mathcal{P}_0^2 + a_0 \mathcal{P}_0 (\mathcal{P}_+^{-\mathbf{k}_F} + \mathcal{P}_+^{+\mathbf{k}_F} + \mathcal{P}_-^{-\mathbf{k}_F} + \mathcal{P}_-^{+\mathbf{k}_F} - 4\mathcal{P}_0) \right. \\ & \left. + a_+ (\mathcal{P}_+^{+\mathbf{k}_F} \mathcal{P}_-^{-\mathbf{k}_F} + \mathcal{P}_+^{-\mathbf{k}_F} \mathcal{P}_-^{+\mathbf{k}_F} - 2\mathcal{P}_0^2) + a_- (\mathcal{P}_-^{-\mathbf{k}_F} \mathcal{P}_-^{+\mathbf{k}_F} + \mathcal{P}_-^{+\mathbf{k}_F} \mathcal{P}_-^{-\mathbf{k}_F} - 2\mathcal{P}_0^2) \right], \end{aligned} \quad (3.42)$$

where

$$a_0 = \frac{4\alpha^2 k_F^2 \Delta^2 \cos^2 \theta_{\mathbf{k}}}{\bar{\Delta}_{\mathbf{k}_F}^2 \bar{\Delta}_{-\mathbf{k}_F}^2}; \quad a_{\pm} = \frac{\alpha^4 k_F^4 + \Delta^4 - 2\alpha^2 k_F^2 \Delta^2 \sin^2 \theta_{\mathbf{k}} \pm \bar{\Delta}_{\mathbf{k}_F} \bar{\Delta}_{-\mathbf{k}_F} (\alpha^2 k_F^2 - \Delta^2)}{\bar{\Delta}_{\mathbf{k}_F}^2 \bar{\Delta}_{-\mathbf{k}_F}^2}. \quad (3.43)$$

The first term in the square brackets in Eq. (3.42) does not depend on the effective field $\bar{\Delta}_{\pm \mathbf{k}_F}$ and can be dropped. The remaining integrals over q are equal to

$$\int dq q \mathcal{P}_0 [\mathcal{P}_+^{-\mathbf{k}_F} + \mathcal{P}_+^{+\mathbf{k}_F} + \mathcal{P}_-^{-\mathbf{k}_F} + \mathcal{P}_-^{+\mathbf{k}_F} - 4\mathcal{P}_0] = \frac{1}{v_F^2} \ln \frac{\Omega^2}{\Omega^2 + \bar{\Delta}_{\mathbf{k}_F}^2} + \frac{1}{v_F^2} \ln \frac{\Omega^2}{\Omega^2 + \bar{\Delta}_{-\mathbf{k}_F}^2}, \quad (3.44)$$

$$\int dq q [\mathcal{P}_+^{+\mathbf{k}_F} \mathcal{P}_-^{-\mathbf{k}_F} + \mathcal{P}_+^{-\mathbf{k}_F} \mathcal{P}_-^{+\mathbf{k}_F} - 2\mathcal{P}_0^2] = \frac{1}{v_F^2} \ln \frac{\Omega^2}{\Omega^2 + (\bar{\Delta}_{\mathbf{k}_F} - \bar{\Delta}_{-\mathbf{k}_F})^2}, \quad (3.45)$$

and

$$\int dq q [\mathcal{P}_-^{-\mathbf{k}_F} \mathcal{P}_-^{+\mathbf{k}_F} + \mathcal{P}_-^{+\mathbf{k}_F} \mathcal{P}_-^{-\mathbf{k}_F} - 2\mathcal{P}_0^2] = \frac{1}{v_F^2} \ln \frac{\Omega^2}{\Omega^2 + (\bar{\Delta}_{\mathbf{k}_F} + \bar{\Delta}_{-\mathbf{k}_F})^2}. \quad (3.46)$$

Evaluating the Matsubara sum in the same way as in Sec. III B, we obtain

$$\begin{aligned} \delta \Xi_{xx}^{(2)} = & -2 \left(\frac{mU}{8\pi v_F} \right)^2 T^3 \int \frac{d\theta_{\mathbf{k}}}{(2\pi)^2} \left\{ a_0 \left[\mathcal{F} \left(\frac{\bar{\Delta}_{\mathbf{k}_F}}{T} \right) + \mathcal{F} \left(\frac{\bar{\Delta}_{-\mathbf{k}_F}}{T} \right) \right] \right. \\ & \left. + a_+ \mathcal{F} \left(\frac{|\bar{\Delta}_{\mathbf{k}_F} - \bar{\Delta}_{-\mathbf{k}_F}|}{T} \right) + a_- \mathcal{F} \left(\frac{\bar{\Delta}_{\mathbf{k}_F} + \bar{\Delta}_{-\mathbf{k}_F}}{T} \right) \right\}, \end{aligned} \quad (3.47)$$

where the function $\mathcal{F}(x)$ and its asymptotic limits are given by Eqs. (3.25-3.27).

The angular integral cannot be performed analytically because the function $\mathcal{F}(y)$ depends in a complicated way on the angle $\theta_{\mathbf{k}}$ through $\bar{\Delta}_{\pm \mathbf{k}_F}$. Therefore, we consider two limiting cases below.

1. Temperature dependence of χ_{xx}

First, we consider the limit of a weak magnetic field: $|\Delta| \ll \max\{|\alpha|k_F, T\}$. The main difference between the cases of in- and transverse orientations of the field is in the term proportional to a_+ in Eq. (3.47). The argument \mathcal{F} in this term vanishes in the limit of $\Delta \rightarrow 0$, whereas the arguments of \mathcal{F} in the rest of the terms reduce to a scaling variable $|\alpha|k_F/T$, as it was also the case for the transverse field. Therefore, the SOI energy scale, $|\alpha|k_F$,

and temperature are interchangeable in the rest of the terms, which means that a nonanalytic T dependence arising from these terms is cut off by the SOI (and vice versa). However, the a_+ term does not depend on α and produces a nonanalytic T dependence which is *not* cut off by the SOI. To see this, we expand prefactors a_0 and a_{\pm} to leading order in Δ as

$$\begin{aligned} a_0 &= 4 \frac{\Delta^2}{\alpha^2 k_F^2} \cos^2 \theta_{\mathbf{k}} + \mathcal{O}(\Delta^4), \\ a_+ &= 2 - 4 \frac{\Delta^2}{\alpha^2 k_F^2} \cos^2 \theta_{\mathbf{k}} + \mathcal{O}(\Delta^4), \\ a_- &= 2 \frac{\Delta^4}{\alpha^4 k_F^4} \cos^4 \theta_{\mathbf{k}} + \mathcal{O}(\Delta^6). \end{aligned} \quad (3.48)$$

The last term in Eq. (3.47), proportional to a_- , does not contribute to order Δ^2 , and we focus on the first two terms. In the term a_0 , we replace $\bar{\Delta}_{\pm \mathbf{k}_F} = |\alpha| k_F$ in the argument of the \mathcal{F} function. In the a_+ term, we replace $|\bar{\Delta}_{\mathbf{k}_F} - \bar{\Delta}_{-\mathbf{k}_F}| = 2|\Delta \sin \theta_{\mathbf{k}}| + \mathcal{O}(\Delta^2)$, and then expand $\mathcal{F}(2|\Delta \sin \theta_{\mathbf{k}}|/T) = 4\Delta^2 \sin^2 \theta_{\mathbf{k}}/T^2$ using Eq. (3.26). Integrating over $\theta_{\mathbf{k}}$ and differentiating the result twice with respect to B , we obtain

$$\begin{aligned} \delta\chi_{xx}^{(2)}(T, \alpha) &= \chi_0 \left(\frac{mU}{4\pi} \right)^2 \left[\frac{T^3}{\alpha^2 k_F^2} \mathcal{F} \left(\frac{|\alpha| k_F}{T} \right) + \frac{T}{E_F} \right] \\ &= \frac{1}{2} \chi_{zz}^{(2)}(T, \alpha) + \frac{1}{2} \chi_0 \left(\frac{mU}{4\pi} \right)^2 \frac{T}{E_F}, \end{aligned} \quad (3.49)$$

where $\chi_{zz}^{(2)}(T, \alpha)$ is the correction to χ_{zz} given by Eq. (3.28). [Notice a remarkable similarity between Eq. (3.49) and the relation between χ_{xx} and χ_{zz} in the non-interacting case, Eq. (A5).] Equation (3.49) is one of the main results of this paper. It shows that a nonanalytic T dependence of χ_{xx} , given by the stand-alone T/E_F term survives in the presence of the SOI. Explicitly, the T dependence is

$$\delta\chi_{xx}^{(2)} = 2\chi_0 \left(\frac{mU}{4\pi} \right)^2 \left[\frac{T}{E_F} + \frac{\alpha^2 k_F^2}{48TE_F} \right] \quad (3.50)$$

for $T \gg |\alpha| k_F$ and

$$\delta\chi_{xx}^{(2)} = 2\chi_0 \left(\frac{mU}{4\pi} \right)^2 \left[\frac{|\alpha| k_F}{6E_F} + \frac{T}{2E_F} + 2\zeta(3) \frac{T^3}{\alpha^2 k_F^2 E_F} \right] \quad (3.51)$$

for $T \ll |\alpha| k_F$.

As it is also the case for the transverse magnetic field, the first term in the second line of Eq. (3.50) is due to particle-hole pairs formed by electrons from three identical and one different Rashba branches. On the other hand, the linear-in- T term, absent in $\delta\chi_{zz}^{(2)}$, is comes from processes involving electrons from the different Rashba branches in each particle-hole bubble, see Fig. 8b. Indeed, pairing electrons and holes, which belong to different Rashba branches and move in the same direction, we obtain particle-hole bubbles $\mathcal{P}_{\pm}^{\pm \mathbf{k}_F}$ [cf. Eq. (3.42)]. The product of two such bubbles, $\mathcal{P}_{+}^{\pm \mathbf{k}_F} \mathcal{P}_{+}^{\mp \mathbf{k}_F}$, being integrated over q and summed over Ω , depends on the difference of the Zeeman energies $|\bar{\Delta}_{\mathbf{k}_F} - \bar{\Delta}_{-\mathbf{k}_F}|$. Since $\sin \theta_{\mathbf{k}}$ is odd upon $\mathbf{k} \rightarrow -\mathbf{k}$, this difference is finite and proportional to $|\Delta|$ for $\Delta \rightarrow 0$ but does not depend on α . This is a mechanism by which one gets an $\mathcal{O}(\Delta^2)$ contribution to the thermodynamic potential and, therefore, a T dependent contribution to χ_{xx} , which does not involve the SOI.

2. Magnetic-field dependence of χ_{xx}

Now we analyze the non-linear in-plane susceptibility $\chi_{xx}(B_x, \alpha) = -\partial^2 \Xi_{xx} / \partial B_x^2$ at $T = 0$. Replacing \mathcal{F} in Eq. (3.47) by its large-argument asymptotics from Eq. (3.27), we obtain

$$\delta\Xi_{xx}^{(2)} = -\frac{2}{3} \left(\frac{mU}{8\pi v_F} \right)^2 \int \frac{d\phi_{kx}}{(2\pi)^2} \left\{ a_0 [\bar{\Delta}_{\mathbf{k}_F}^3 + \bar{\Delta}_{-\mathbf{k}_F}^3] + a_+ |\bar{\Delta}_{\mathbf{k}_F} - \bar{\Delta}_{-\mathbf{k}_F}|^3 + a_- [\bar{\Delta}_{\mathbf{k}_F} + \bar{\Delta}_{-\mathbf{k}_F}]^3 \right\}. \quad (3.52)$$

The angular integral can now be solved explicitly in the limiting cases of $|\Delta| \ll |\alpha| k_F$ and $|\Delta| \gg |\alpha| k_F$. Since our primary interest is just to see whether a nonanalytic field-dependence survives in the presence of the SOI, we will consider only the weak-field case: $|\Delta| \ll |\alpha| k_F$. The a_- term in Eq. (3.52) can then be dropped, while the a_0

and a_+ ones yield

$$\delta\chi_{xx}^{(2)}(B_x, \alpha) = \frac{1}{3} \chi_0 \left(\frac{mU}{4\pi} \right)^2 \left[\frac{|\alpha| k_F}{E_F} + \frac{16 |\Delta|}{\pi E_F} \right]. \quad (3.53)$$

The first term in Eq. (3.53) is just half of the first term in $\delta\chi_{zz}^{(2)}$ [cf. Eq. (3.37)]. However, the second term represents a nonanalytic dependence on the field which is not cut off by the SOI.

D. Remaining second order diagrams

Besides the diagrams considered so far, there are other second order diagrams, which – in principle – could contribute to the spin susceptibility. These diagrams are depicted in Fig. 6*b-e*. In the absence of the SOI, these diagrams are irrelevant because the electron-electron interaction conserves spin. This means that the spins of electrons in each of the bubbles in, e.g., diagram *b*) are the same and, therefore, the Zeeman energies, entering the Green's functions, can be absorbed into the chemical potential. The same argument also goes for the other two diagrams. In the presence of the SOI, this argument does not work because spin is not a good quantum number and the interaction mixes states from all Rashba branches with different Zeeman energies. However, one can show that the net result is the same as without the SOI: diagrams in Fig. 6*b-e* do not contribute to the non-analytic behavior of the spin susceptibility. This is what we are going to show in this section.

We begin with diagram *b*), which is a small momentum-transfer counterpart of diagram *a*):

$$\delta\Xi_b^{(2)} = -U^2(0)T \sum_Q \left[\text{Tr} \hat{\Pi}(Q) \right]^2, \quad (3.54)$$

where

$$\begin{aligned} \hat{\Pi}(Q) &= \frac{1}{2} \sum_K \hat{G}(K) \hat{G}(K+Q) \\ &= \frac{1}{2} \sum_{s,t=\pm 1} \hat{\Omega}_s \hat{\Omega}_t \sum_K g_s(K) g_t(K+Q) \end{aligned} \quad (3.55)$$

is the full (summed over Rashba branches) polarization bubble, and both the space- and time-like components of $Q \equiv (\Omega, \mathbf{q})$ are small. As it is also the case in the absence of the SOI, the small- Q bubble does not depend on the magnetic field. Indeed, noticing that the matrix

$\hat{\zeta}$ in Eq. (2.2a) has the following properties

$$\hat{\zeta}^2 = \hat{I} \text{ and } \text{Tr} \hat{\zeta} = 0, \quad (3.56)$$

it is easy to show that

$$\hat{\Omega}_s \hat{\Omega}_t = \frac{1}{4} \left[(1+st)\hat{I} + (s+t)\hat{\zeta} \right]. \quad (3.57)$$

Consequently, $\hat{\Omega}_+ \hat{\Omega}_+ = \hat{\Omega}_- \hat{\Omega}_- = (1/2)\hat{I}$ and $\hat{\Omega}_+ \hat{\Omega}_- = \hat{\Omega}_- \hat{\Omega}_+ = 0$. Therefore, electrons from different branches do not contribute to $\hat{\Pi}(Q)$, while the Zeeman energies in the contributions from the same branch can be absorbed into the chemical potential. As result, the dynamic part of the bubble depends neither on the field nor on the SOI (as long as a weak dependence of the Fermi velocity for a given branch on α is neglected):

$$\hat{\Pi}(Q) = \hat{I} \frac{m}{2\pi} \frac{|\Omega|}{\sqrt{\Omega^2 + v_F^2 q^2}}. \quad (3.58)$$

Therefore diagram *b*) does not contribute to the spin susceptibility. We remind the reader that, since there are no threshold-like singularities in the static polarization bubble (see discussion in Sec. III), the Landau-damping singularity in Eq. (3.58) is the only singularity which may have contributed to a nonanalytic behavior of the spin susceptibility. However, as we have just demonstrated, Landau damping is not effective in diagrams with small momentum transfers.

Similarly, diagram *c*) with two crossed interaction lines, one of which carries a small momentum and the other one carries a momentum near $2k_F$, is expressed via a small Q bubble as

$$\delta\Xi_c^{(2)} = -U(0)U(2k_F) \sum_Q \text{Tr} \left[\hat{\Pi}^2(Q) \right] \quad (3.59)$$

and, therefore, does not depend on the magnetic field. Diagram *d*), with both interaction lines carrying small momenta, contains a trace of four Green's functions

$$\delta\Xi_d^{(2)} = -\frac{U^2(0)}{4} \sum_{Q,Q',K} \text{Tr} \left[\hat{G}(K) \hat{G}(K+Q) \hat{G}(K+Q+Q') \hat{G}(K+Q') \right] \quad (3.60)$$

$$= -\frac{U^2(0)}{4} \sum_{Q,Q',K} \sum_{p,r,s,t=\pm} \text{Tr} \left[\hat{\Omega}_p \hat{\Omega}_r \hat{\Omega}_s \hat{\Omega}_t \right] g_p(K) g_r(K+Q) g_s(K+Q+Q') g_t(K+Q'), \quad (3.61)$$

where Q and Q' are small, so that dependence of $\hat{\Omega}_l$ on either of the bosonic momenta can be neglected. Us-

ing again the properties of the projection operator from Eq. (3.56), we find that

$$\begin{aligned} \text{Tr} \left[\hat{\Omega}_p \hat{\Omega}_r \hat{\Omega}_s \hat{\Omega}_t \right] &= \frac{1}{16} \text{Tr} \left[\{ (1+pr)(1+st) + (p+r)(s+t) \} \hat{I} + \{ (1+pr)(s+t) + (1+st)(p+r) \} \hat{\zeta} \right] \\ &= \frac{1}{8} [(1+pr)(1+st) + (p+r)(s+t)]. \end{aligned} \quad (3.62)$$

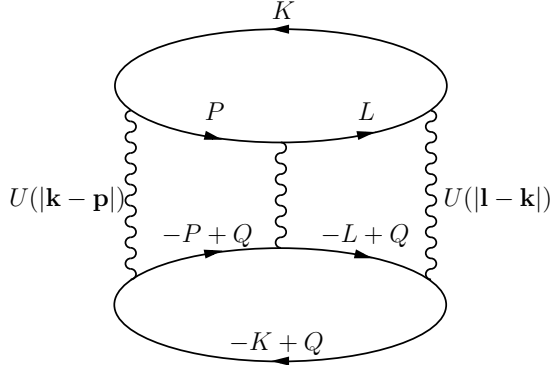


FIG. 9. The third-order Cooper-channel diagram for the thermodynamic potential.

This expression vanishes if at least one of the indices from the set $\{p, r, s, t\}$ is different from the others. Therefore, only electrons from the same branch contribute to $\delta\Xi_d^{(2)}$, the Zeeman energy can again be absorbed into the chemical potential, and $\delta\Xi_d^{(2)}$ does not depend on the magnetic field.

Finally, the last diagram, $\delta\Xi_e^{(2)}$ corresponds to the first-order self-energy inserted twice into the zeroth order thermodynamic potential. Such an insertion only shifts the chemical potential and, for a q dependent U , gives a regular correction to the electron effective mass but does not produce any nonanalytic behavior.

IV. COOPER-CHANNEL RENORMALIZATION

A. General remarks

The second-order nonanalytic contribution to the spin susceptibility comes from “backscattering” processes in which two fermions moving in almost opposite directions experience almost complete backscattering. Because the total momentum of two fermions is small, the backscattering process is a special case of the interaction in the Cooper (particle-particle) channel. Higher order processes in this channel lead to logarithmic renormalization of the second-order result.^{3,6–8,10,19–22,29,30} For a weak interaction, considered throughout this paper, this is the leading higher-order effect. In the absence of the SOI, resummation of all orders in the Cooper channel leads to the following scaling form of the spin susceptibility

$$\delta\chi \propto \frac{E}{\ln^2(E/\Lambda)}, \quad (4.1)$$

where Λ is the ultraviolet cutoff and $E \equiv \max\{T, |\Delta|\}$ is small enough so that $mU|\ln(E/\Lambda)| \gg 1$. As we see, both the linear-in- E term, which occurs already at second order, and its logarithmic renormalization contain

the same energy scale. The reason for this symmetry is very simple: an arbitrary order Cooper diagram for the thermodynamic potential contains two bubbles joined by a ladder. The spin susceptibility is determined only by diagrams with opposite fermion spins in each of the bubbles. Therefore, all Cooper bubbles in such diagrams are formed by fermions with opposite spins, so that the logarithmic singularity of the bubble is cut off at the largest of the two energy scales, i.e., temperature or Zeeman energy. At zero incoming momentum and frequency, the Cooper bubble is

$$\Pi_C^{\uparrow\downarrow} = \frac{m}{2\pi} \ln \frac{\Lambda}{\max\{T, \Delta\}}, \quad (4.2)$$

hence the symmetry of the result with respect to interchanging T and Δ follows immediately. It will be shown in this section that this symmetry does not hold in the presence of the SOI. The reason is that the Rashba branches are not the states with definite spins, and diagrams with Cooper bubbles formed by electrons from the same branches also contribute to the spin susceptibility. Although a Cooper bubble formed by electrons from branches s and s'

$$\begin{aligned} \Pi_C^{s,s'} &= T \sum_{\omega} \frac{m}{2\pi} \int \frac{d\theta_{\mathbf{p}}}{2\pi} \int d\xi_p g_s(\omega, \mathbf{p}) g_{s'}(-\omega, -\mathbf{p}) \\ &= \frac{m}{2\pi} \ln \frac{\Lambda}{\max\{T, |s - s'| \Delta_{\mathbf{k}_F}\}} \end{aligned} \quad (4.3)$$

looks similar to that in the absence of the SOI [Eq. (4.2)], its diagonal element Π_C^{ss} depends only on T even if $T \ll \Delta_{\mathbf{k}_F}$. Therefore, for $\Delta = 0$, the Cooper logarithm in Eq. (4.1) will depend only on T in the limit of $T \rightarrow 0$ while the energy E in the numerator may be given either by T or by $|\alpha|k_F$.

B. Third-order Cooper channel contribution to χ_{zz}

In this section, we obtain the third-order Cooper channel contribution to χ_{zz} . This calculation will help to understand the general strategy employed later, in Secs. IV C 3 and IV C 4, in resumming Cooper diagrams to all orders.

The third-order Cooper diagram for the thermodynamic potential, depicted in Fig. 9, is given by

$$\begin{aligned} \delta\Xi_{zz}^{(3)} &= \frac{U^3}{6} \sum_{K,P,L,Q} \text{Tr}(\hat{G}_K \hat{G}_P \hat{G}_L) \\ &\quad \times \text{Tr}(\hat{G}_{-K+Q} \hat{G}_{-P+Q} \hat{G}_{-L+Q}). \end{aligned} \quad (4.4)$$

First, we evaluate the traces $\text{Tr}(\hat{G}_K \hat{G}_P \hat{G}_L) = \sum_{rst} \mathcal{B}_{rst} g_r(K) g_s(P) g_t(L)$ with the coefficients

$$\mathcal{B}_{rst} \equiv \text{Tr}[\hat{\Omega}_r(\mathbf{k})\hat{\Omega}_s(\mathbf{p})\hat{\Omega}_t(\mathbf{l})] = \frac{1}{4\bar{\Delta}_{\mathbf{k}}\bar{\Delta}_{\mathbf{p}}\bar{\Delta}_{\mathbf{l}}} [\bar{\Delta}_{\mathbf{k}}\bar{\Delta}_{\mathbf{p}}\bar{\Delta}_{\mathbf{l}} + i r s t \alpha^2 \Delta(\mathbf{k} \times \mathbf{p} + \mathbf{p} \times \mathbf{l} + \mathbf{l} \times \mathbf{k})_z \\ + r s (\alpha^2 \mathbf{k} \cdot \mathbf{p} + \Delta^2) \bar{\Delta}_{\mathbf{l}} + r t (\alpha^2 \mathbf{k} \cdot \mathbf{l} + \Delta^2) \bar{\Delta}_{\mathbf{p}} + s t (\alpha^2 \mathbf{l} \cdot \mathbf{p} + \Delta^2) \bar{\Delta}_{\mathbf{k}}]. \quad (4.5)$$

Since \mathbf{q} is small and \mathcal{B}_{rst} is an even function of the fermionic momenta,

$$\text{Tr} [\hat{\Omega}_{r'}(-\mathbf{k} + \mathbf{q})\hat{\Omega}_{s'}(-\mathbf{p} + \mathbf{q})\hat{\Omega}_{t'}(-\mathbf{l} + \mathbf{q})] \\ \approx \text{Tr} [\hat{\Omega}_{r'}(-\mathbf{k})\hat{\Omega}_{s'}(-\mathbf{p})\hat{\Omega}_{t'}(-\mathbf{l})] = \mathcal{B}_{r's't'}, \quad (4.6)$$

and the thermodynamic potential becomes

$$\delta\Xi_{zz}^{(3)} = \frac{U^3}{6} \sum_{K,P,L,Q} \sum_{rst} \sum_{r's't'} \mathcal{B}_{rst} \mathcal{B}_{r's't'} g_r(K) g_{r'}(-K+Q) \\ \times g_s(P) g_{s'}(-P+Q) g_t(l) g_{t'}(-L+Q). \quad (4.7)$$

Each pair of the Green's functions with opposite momenta forms a Cooper bubble, which depends logarithmically on the largest of the two energy scales—temperature

or the effective Zeeman energy, see Eq. (4.3). The third-order contribution contains one such logarithmic factor which can be extracted from any of the three Cooper bubbles; this gives an overall factor of three:

$$\delta\Xi_{zz}^{(3)} = \frac{U^3}{2} \sum_{K,P,Q} \sum_{rst} \sum_{r's't'} \int \frac{d\theta_{\mathbf{kl}}}{2\pi} \mathcal{B}_{rst} \mathcal{B}_{r's't'} \Pi_C^{t,t'} \\ \times g_r(K) g_{r'}(-K+Q) g_s(P) g_{s'}(-P+Q) \quad (4.8)$$

where $\theta_{\mathbf{kl}} \equiv \angle(\mathbf{k}, \mathbf{l})$. With this procedure, the third-order diagram reduces effectively to the second-order one, but with a new set of coefficients. Since we already know that the nonanalytic part of the second-order diagram comes from processes with $\mathbf{k} \approx -\mathbf{p}$, the coefficient \mathcal{B}_{rst} (and its primed counterpart) simplify significantly because $\mathbf{k} \times \mathbf{p} = \mathbf{0}$ and $\mathbf{p} \times \mathbf{l} = \mathbf{l} \times \mathbf{k}$. Integrating over $\theta_{\mathbf{kl}}$, we obtain

$$\mathcal{A}_{rst r' s' t'} \equiv \int \frac{d\theta_{\mathbf{kl}}}{2\pi} \mathcal{B}_{rst} \mathcal{B}_{r' s' t'} = \frac{1}{16\bar{\Delta}_{\mathbf{k}_F}^6} \left\{ -2rst r' s' t' \alpha^4 k_F^4 \Delta^2 + \frac{1}{2} (rtr't' + sts't' - rts't' - r't'st) \alpha^4 k_F^4 \bar{\Delta}_{\mathbf{k}_F}^2 \right. \\ \left. + \bar{\Delta}_{\mathbf{k}_F}^2 [\bar{\Delta}_{\mathbf{k}_F}^2 + (rs + rt + st) \Delta^2 - rs \alpha^2 k_F^2] [\bar{\Delta}_{\mathbf{k}_F}^2 + (r's' + r't' + s't') \Delta^2 - r's' \alpha^2 k_F^2] \right\}, \quad (4.9)$$

and

$$\delta\Xi_{zz}^{(3)} = \frac{U^3}{2} \sum_{rst} \sum_{r's't'} \mathcal{A}_{rst r' s' t'} \Pi_C^{t,t'} \sum_{K,P,Q} g_r(K) g_{r'}(-K+Q) g_s(P) g_{s'}(-P+Q). \quad (4.10)$$

The convolution of two Cooper bubbles in Eq. (4.10) can be re-written via a convolution of two particle-hole bubbles by relabeling $Q \rightarrow Q + K + P$:

$$\sum_{K,P,Q} g_r(K) g_{r'}(-K+Q) g_s(P) g_{s'}(-P+Q) = \sum_Q \sum_K g_r(K) g_{s'}(K+Q) \sum_P g_s(P) g_{r'}(P+Q) \sum_Q \Pi_{rs'}(Q) \Pi_{sr'}(Q) \quad (4.11)$$

where $\Pi_{s,s'}(Q)$ is a particle-hole bubble defined in Eq. (3.6). Summing over the Rashba indices, integrating over the momentum, and assuming that the Zeeman energy is the smallest energy in the problem, i.e., that $\Delta \ll \max\{T, |\alpha|k_F\}$, we find

$$\delta\Xi_{zz}^{(3)} = -\frac{1}{2\pi v_F^2} \left(\frac{mU}{4\pi} \right)^3 \frac{\Delta^2 T^3}{\alpha^2 k_F^2} \left\{ \left[12\mathcal{F}\left(\frac{|\alpha|k_F}{T}\right) - \mathcal{F}\left(\frac{2|\alpha|k_F}{T}\right) \right] \ln \frac{T}{\Lambda} \right. \\ \left. + \left[4\mathcal{F}\left(\frac{|\alpha|k_F}{T}\right) + \mathcal{F}\left(\frac{2|\alpha|k_F}{T}\right) \right] \ln \frac{\max\{T, |\alpha|k_F\}}{\Lambda} \right\} \quad (4.12)$$

with $\mathcal{F}(y)$ given by Eq. (3.25).

The asymptotic behavior of χ_{zz} for $T \gg |\alpha|k_F$ is com-

puted from Eq. (4.12)

$$\delta\chi_{zz}^{(3)} = 8\chi_0 \left(\frac{mU}{4\pi} \right)^3 \frac{T}{E_F} \ln \frac{T}{\Lambda} \quad (4.13)$$

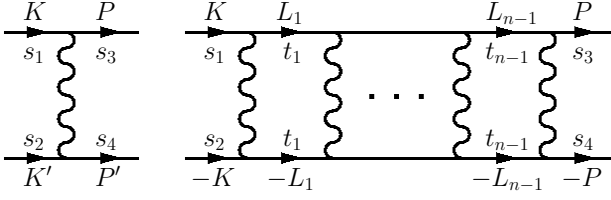


FIG. 10. Left: the effective scattering amplitude $\Gamma_{s_1 s_2; s_3 s_4}^{(1)}(\mathbf{k}, \mathbf{k}'; \mathbf{p}, \mathbf{p}')$ in the chiral basis. Right: a generic n -th order ladder diagram in the Cooper channel, $\Gamma_{s_1 s_2; s_3 s_4}^{(n)}(\mathbf{k}, -\mathbf{k}; \mathbf{p}, -\mathbf{p})$.

and, as to be expected, $\delta\chi_{zz}^{(3)}$ scales as $T \ln T$.

In the opposite limit of $T \ll |\alpha|k_F$,

$$\begin{aligned} \delta\chi_{zz}^{(3)} &= \frac{2}{3}\chi_0 \left(\frac{mU}{4\pi} \right)^3 \frac{|\alpha|k_F}{E_F} \left(\ln \frac{T}{\Lambda} + 3 \ln \frac{|\alpha|k_F}{\Lambda} \right) \\ &\approx \frac{2}{3}\chi_0 \left(\frac{mU}{4\pi} \right)^3 \frac{|\alpha|k_F}{E_F} \ln \frac{T}{\Lambda}, \end{aligned} \quad (4.14)$$

since $|\ln(T/\Lambda)| \gg |\ln(|\alpha|k_F/\Lambda)|$. As it was advertised in Sec. IV A, the $T \ln T$ scaling at high temperatures is replaced by the $|\alpha| \ln T$ scaling at low temperatures which implies that the energy scales $|\alpha|k_F$ and T are not interchangeable.

C. Re-summation of all diagrams in the Cooper channel

1. Scattering amplitude in the chiral basis

It is more convenient to resum the Cooper ladder diagrams in the chiral basis, in which the Green's functions are diagonal. Introducing Rashba spinors $|\mathbf{k}, s\rangle$, we rewrite the number-density operator as

$$\hat{\rho}_{\mathbf{q}} = \sum_{\mathbf{k}} \sum_{s_1, s_2} \langle \mathbf{k} + \mathbf{q}, s_2 | \mathbf{k}, s_1 \rangle \hat{c}_{\mathbf{k}+\mathbf{q}, s_2}^\dagger \hat{c}_{\mathbf{k}, s_1}, \quad (4.15)$$

so that the Hamiltonian of the four-fermion interaction becomes

$$\hat{H}_{\text{int}} = \frac{1}{2} \sum_{\mathbf{q}} U(\mathbf{q}) \hat{\rho}_{\mathbf{q}} \hat{\rho}_{-\mathbf{q}} = \frac{1}{2} \sum_{\mathbf{p}, \mathbf{p}', \mathbf{k}, \mathbf{k}'} \sum_{\{s_i\}} \Gamma_{s_1 s_2; s_3 s_4}^{(1)}(\mathbf{k}, \mathbf{k}'; \mathbf{p}, \mathbf{p}') \hat{c}_{\mathbf{p}', s_4}^\dagger \hat{c}_{\mathbf{p}, s_3}^\dagger \hat{c}_{\mathbf{k}, s_1} \hat{c}_{\mathbf{k}', s_2}, \quad (4.16)$$

where the effective scattering amplitude is defined by (cf. Fig. 10)

$$\Gamma_{s_1 s_2; s_3 s_4}^{(1)}(\mathbf{k}, \mathbf{k}'; \mathbf{p}, \mathbf{p}') = U(\mathbf{k} - \mathbf{p}) \langle \mathbf{p}, s_3 | \mathbf{k}, s_1 \rangle \langle \mathbf{p}', s_4 | \mathbf{k}', s_2 \rangle. \quad (4.17)$$

In the absence of the magnetic field,

$$|\mathbf{k}, s\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -ise^{-i\theta_{\mathbf{k}}} \\ 1 \end{pmatrix}, \quad (4.18)$$

where $\theta_{\mathbf{k}} \equiv \angle(\mathbf{k}, \hat{x})$, and Eq. (4.17) gives³¹

$$\begin{aligned} \Gamma_{s_1 s_2; s_3 s_4}^{(1)}(\mathbf{k}, \mathbf{k}'; \mathbf{p}, \mathbf{p}') &= \frac{1}{4} U(\mathbf{k} - \mathbf{p}) \left[1 + s_1 s_3 e^{i(\theta_{\mathbf{p}} - \theta_{\mathbf{k}})} \right] \\ &\times \left[1 + s_2 s_4 e^{i(\theta_{\mathbf{p}'} - \theta_{\mathbf{k}'})} \right]. \end{aligned} \quad (4.19)$$

In order to resum the ladder diagrams for the thermodynamic potential to infinite order, we consider a skeleton diagram depicted in Fig. 11, which is obtained from the second-order diagram –shown in Fig. 6a– by replacing the bare interaction $U(\mathbf{q})$ with the dressed scattering amplitudes: $\Gamma_{s_1 s_2; s_3 s_4}(\mathbf{k}, -\mathbf{k} + \mathbf{q}; \mathbf{p}, -\mathbf{p} + \mathbf{q})$ and its time reversed counterpart. The dressed amplitudes contain infinite sums of the Cooper ladder diagrams shown in Fig. 10. We will be interested in the limit of vanishingly small magnetic fields and temperatures smaller than the SOI energy scale: $\Delta \ll T \ll |\alpha|k_F$. In this limit, the

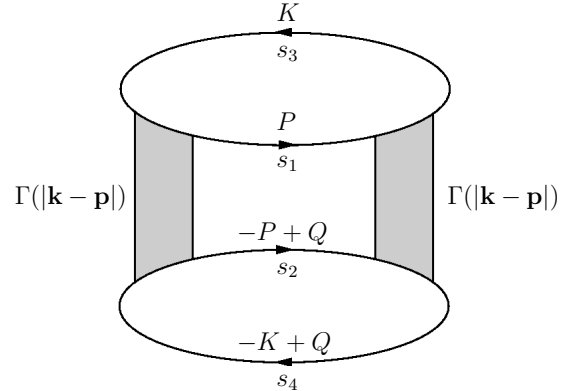


FIG. 11. A skeleton diagram for the thermodynamic potential Ξ .

largest contribution to the ladder diagrams comes from the internal Cooper bubbles formed by electrons from the same Rashba subbands. Each "rung" of this ladder contributes a large Cooper logarithm $L \equiv (m/2\pi) \ln(\Lambda/T)$, which depends only on the temperature, and one has to select the diagrams with a maximum number of L factors.

2. Renormalization group for scattering amplitudes

Resummation of Cooper diagrams is performed most conveniently via the Renormalization Group (RG) procedure⁸. In the Cooper channel, the bare amplitude is given by Eq. (4.19) with $\mathbf{p}' = -\mathbf{p}$ and $\mathbf{k}' = -\mathbf{k}$ or, equivalently, $\theta_{\mathbf{k}'} = \theta_{\mathbf{k}} + \pi$ and $\theta_{\mathbf{p}'} = \theta_{\mathbf{p}} + \pi$:

$$\Gamma_{s_1 s_2; s_3 s_4}^{(1)}(\mathbf{k}, -\mathbf{k}; \mathbf{p}, -\mathbf{p}) = U_{s_1 s_2; s_3 s_4}^{(1)} + V_{s_1 s_2; s_3 s_4}^{(1)} e^{i(\theta_{\mathbf{p}} - \theta_{\mathbf{k}})} + W_{s_1 s_2; s_3 s_4}^{(1)} e^{2i(\theta_{\mathbf{p}} - \theta_{\mathbf{k}})}, \quad (4.20)$$

where the three terms correspond to orbital momenta $\ell = 0, 1, 2$, respectively. The bare values of partial amplitudes are given by

$$U_{s_1 s_2; s_3 s_4}^{(1)} = U/4, \quad (4.21a)$$

$$V_{s_1 s_2; s_3 s_4}^{(1)} = (U/4)(s_1 s_3 + s_2 s_4), \quad (4.21b)$$

$$W_{s_1 s_2; s_3 s_4}^{(1)} = (U/4)s_1 s_2 s_3 s_4. \quad (4.21c)$$

Consider now a ladder diagram consisting of n interaction lines and $2(n-1)$ internal fermionic lines, as shown in Fig. 10. As we have already pointed out, in the limit $T \ll |\alpha|k_F$, the dominant logarithmic-in- T contribution originates from those Cooper bubbles which are formed by electrons from the same Rashba branch. Therefore, the n -th order Cooper ladder can be written iteratively as

$$\begin{aligned} \Gamma_{s_1 s_2; s_3 s_4}^{(n)}(\mathbf{k}, -\mathbf{k}; \mathbf{p}, -\mathbf{p}) &= \\ &= -L \int_{\theta_1} \sum_s \Gamma_{s_1 s_2; ss}^{(n-1)}(\mathbf{k}, -\mathbf{k}; \mathbf{l}, -\mathbf{l}) \Gamma_{ss; s_3 s_4}^{(1)}(\mathbf{l}, -\mathbf{l}; \mathbf{p}, -\mathbf{p}) \end{aligned} \quad (4.22)$$

where $n \geq 2$. Since only “charge-neutral” terms of the type $e^{i(\theta_{\mathbf{p}} - \theta_{\mathbf{l}})} e^{i(\theta_{\mathbf{l}} - \theta_{\mathbf{k}})}$ survive upon averaging over $\theta_{\mathbf{l}}$, different partial harmonics are renormalized independently of each other, i.e., we have the following group property:

$$\Gamma_{s_1 s_2; s_3 s_4}^{(n)}(\mathbf{k}, -\mathbf{k}; \mathbf{p}, -\mathbf{p}) = (-L)^{n-1} [U_{s_1 s_2; s_3 s_4}^{(n)} + V_{s_1 s_2; s_3 s_4}^{(n)} e^{i(\theta_{\mathbf{p}} - \theta_{\mathbf{k}})} + W_{s_1 s_2; s_3 s_4}^{(n)} e^{2i(\theta_{\mathbf{p}} - \theta_{\mathbf{k}})}]. \quad (4.23)$$

Differentiating Eq. (4.22) for $n = 2$ with respect to L we obtain three decoupled one-loop RG flow equations

$$-\frac{d}{dL} U_{s_1 s_2; s_3 s_4}(L) = \sum_s U_{s_1 s_2; ss}(L) U_{ss; s_3 s_4}(L), \quad (4.24a)$$

$$-\frac{d}{dL} V_{s_1 s_2; s_3 s_4}(L) = \sum_s V_{s_1 s_2; ss}(L) V_{ss; s_3 s_4}(L), \quad (4.24b)$$

$$-\frac{d}{dL} W_{s_1 s_2; s_3 s_4}(L) = \sum_s W_{s_1 s_2; ss}(L) W_{ss; s_3 s_4}(L) \quad (4.24c)$$

with initial conditions specified by $U_{s_1 s_2; s_3 s_4}(0) = U_{s_1 s_2; s_3 s_4}^{(1)}$, $V_{s_1 s_2; s_3 s_4}(0) = V_{s_1 s_2; s_3 s_4}^{(1)}$, and $W_{s_1 s_2; s_3 s_4}(0) = W_{s_1 s_2; s_3 s_4}^{(1)}$.

Solving this system of RG equations and substituting the results into the backscattering amplitude,

$$\Gamma_{s_1 s_2; s_3 s_4}(\mathbf{k}, -\mathbf{k}; -\mathbf{k}, \mathbf{k}) = U_{s_1 s_2; s_3 s_4}(L) - V_{s_1 s_2; s_3 s_4}(L) + W_{s_1 s_2; s_3 s_4}(L) \quad (4.25)$$

which is a special case of the Cooper amplitude for $\mathbf{p} = -\mathbf{k}$, we obtain

$$\Gamma_{ss; \pm s \pm s}(\mathbf{k}, -\mathbf{k}; -\mathbf{k}, \mathbf{k}) = \frac{U}{2+UL} \mp \frac{U}{2(1+UL)}, \quad (4.26)$$

$$\Gamma_{s-s; \mp s \pm s}(\mathbf{k}, -\mathbf{k}; -\mathbf{k}, \mathbf{k}) = \frac{U}{2+UL} \pm \frac{U}{2}, \quad (4.27)$$

$$\Gamma_{\sigma(\pm \mp; \mp \mp)}(\mathbf{k}, -\mathbf{k}; -\mathbf{k}, \mathbf{k}) = 0, \quad (4.28)$$

where $\sigma(s_1 s_2; s_3 s_4) \equiv \{(s_1 s_2; s_3 s_4), (s_2, s_3; s_4 s_1), (s_3, s_4; s_1 s_2), (s_4, s_1; s_2, s_3)\}$ stands for all cyclic permutations of indices.

We see that the RG flow described by Eqs. (4.24a-4.24c) has a non-trivial solution: whereas the amplitudes in Eqs. (4.26) and (4.28) flow to zero in the limit of $L \rightarrow \infty$, the amplitudes in Eq. (4.27) approach RG-invariant values of $\pm U/2$. This behavior is in a striking contrast to what one finds in the absence of the SOI, when the repulsive interaction is renormalized to zero in the Cooper channel. Notice that we consider only the energy scales below the SOI energy, while the conventional behavior is recovered at energies above the SOI scale.

The scattering amplitudes can be also derived by iterating Eq. (4.22) directly. Examining a few first orders, one recognizes the pattern for the n -th order partial amplitudes to be

$$\Gamma_{ss; \pm s \pm s}^{(n)}(\mathbf{k}, -\mathbf{k}; -\mathbf{k}, \mathbf{k}) = U^n (-L)^{n-1} \frac{1 \mp 2^{n-1}}{2^n}, \quad (4.29)$$

$$\Gamma_{s-s; \mp s \pm s}^{(n)}(\mathbf{k}, -\mathbf{k}; -\mathbf{k}, \mathbf{k}) = U^n (-L)^n \left(\frac{1}{2^n} \pm \frac{1}{2} \delta_{n,1} \right), \quad (4.30)$$

$$\Gamma_{\sigma(\pm \mp; \mp \mp)}^{(n)}(\mathbf{k}, -\mathbf{k}; -\mathbf{k}, \mathbf{k}) = 0. \quad (4.31)$$

Summing these amplitudes over n , one reproduces the RG result.

3. Renormalization of χ_{zz}

The infinite-order result for the thermodynamic potential is obtained by replacing the bare contact interaction U by its “dressed” counterpart Γ in the second-order skeleton diagram Fig. 11

$$\begin{aligned} \delta \Xi_{zz} &= -\frac{1}{4} \sum_Q \sum_{\{s_i\}} \Gamma_{s_1 s_4; s_3 s_2}(\mathbf{k}, -\mathbf{k}; -\mathbf{k}, \mathbf{k}) \\ &\quad \times \Gamma_{s_3 s_2; s_1 s_4}(-\mathbf{k}, \mathbf{k}; \mathbf{k}, -\mathbf{k}) \Pi_{s_1 s_2} \Pi_{s_3 s_4} \end{aligned} \quad (4.32)$$

with $\Gamma_{s_3 s_4; s_1 s_2}(-\mathbf{k}, \mathbf{k}; \mathbf{k}, -\mathbf{k}) = \Gamma_{s_1 s_2; s_3 s_4}(\mathbf{k}, -\mathbf{k}; -\mathbf{k}, \mathbf{k})$.

Now, we derive the asymptotic form of $\delta\chi_{zz}$ valid in the limit of strong Cooper renormalization, i.e., for $UL \gg 1$. In this limit, the only non-vanishing scattering amplitude is given by Eq. (4.27). Replacing the full Γ by its RG-invariant asymptotic limit $\Gamma_{s-s; \mp s \pm s}(\mathbf{k}, -\mathbf{k}; -\mathbf{k}, \mathbf{k}) = \pm U/2$, we obtain

$$\delta\Xi_{zz} = -\frac{U^2}{16} T \sum_{\Omega} \int \frac{q dq}{2\pi} [(\Pi_{+-}^2 + \Pi_{-+}^2 - 2\Pi_0^2) + 4\Pi_0^2] \quad (4.33)$$

In contrast to the perturbation theory, where the magnetic-field dependence of the thermodynamic potential was provided by the vertices while the polarization bubbles supplied the dependence on the temperature and on the SOI, the vertices in the non-perturbative result (4.33) depend neither on the field nor on the SOI. Therefore, the dependences of Ξ on all three parameters (B , T , and α) must come from the polarization bubbles. The integral over q along with the sum over the Matsubara frequency Ω have already been performed in Sec. III. Note that the last term in square brackets (proportional to Π_0^2) does not depend on the magnetic field and thus

can be dropped. The final result reads

$$\delta\Xi_{zz} = -\frac{T^3}{8\pi v_F^2} \left(\frac{mU}{2\pi}\right)^2 \mathcal{F}\left(\frac{2\bar{\Delta}_{\mathbf{k}_F}}{T}\right) \quad (4.34)$$

so that

$$\delta\chi_{zz} = \frac{\chi_0}{2} \left(\frac{mU}{4\pi}\right)^2 \frac{|\alpha|k_F}{E_F}, \quad (4.35)$$

where use was made of the expansion

$$\frac{\partial^2}{\partial \Delta^2} \mathcal{F}\left(\frac{2\bar{\Delta}_{\mathbf{k}_F}}{T}\right) \approx \frac{2}{|\alpha|k_F T} \mathcal{F}'\left(\frac{2|\alpha|k_F}{T}\right) \quad (4.36)$$

and made use of the asymptotic form (3.27) of the function \mathcal{F} to find that $\mathcal{F}'(x) \approx x^2$ for $x \gg 1$. Comparing the non-perturbative and second-order results for χ_{zz} [given by Eqs. (4.35) and (3.30), respectively], we see that the only effect of Cooper renormalization is a change in the numerical coefficient of the nonanalytic part of χ_{zz} . This is a consequence of a non-trivial fixed point in the Cooper channel which corresponds to finite rather than vanishing Coulomb repulsion.

The temperature dependence of χ_{zz} can be also found for an arbitrary value of the Cooper renormalization parameter UL . Deferring the details to Appendix C1, we present here only the final result

$$\begin{aligned} \delta\chi_{zz} = \chi_0 \frac{2|\alpha|k_F}{E_F} \left(\frac{mU}{4\pi}\right)^2 & \left[\left(\frac{1}{2+UL} - \frac{1}{2}\right)^2 + \frac{1}{3} \left(\frac{1}{2(1+UL)} + \frac{1}{2+UL}\right)^2 \right. \\ & \left. + \frac{4}{3} \left(\frac{1}{2(1+UL)} - \frac{2}{(2+UL)^2} + \frac{2}{2+UL}\right) \left(\frac{1}{2+UL} - \frac{1}{2}\right) \right]. \end{aligned} \quad (4.37)$$

In the limit of strong renormalization in the Cooper channel, i.e., for $UL \gg 1$, only the first term survives the logarithmic suppression, and Eq. (4.37) reduces to Eq. (4.35). In the absence of Cooper renormalization, i.e., for $L = 0$, Eq. (4.37) reduces to the second-order result (3.30). In between these two limits, $\delta\chi_{zz}$ is a non-monotonic function of UL : as shown in Fig. 12, $\delta\chi_{zz}$ exhibits a (shallow) minimum at $UL \approx 2.1$. In a wide interval of UL ($0.9 \leq UL \leq 5.6$), the sign of $\delta\chi_{zz}$ is opposite (negative) to that in either of the high- and low-temperature limits. It is also seen from this plot that the low- T asymptotic value (marked by a straight line) is reached only at very large ($\gtrsim 100$) values of UL .

4. Renormalization of χ_{xx}

The second-order result for the in-plane magnetic field is renormalized in a similar way with two exceptions. First, because the Zeeman energy is anisotropic

in this case – the effective magnetic field $\bar{\Delta}_{\pm\mathbf{k}_F}(\theta_1) \equiv \sqrt{\alpha^2 k_F^2 \pm 2\alpha k_F \Delta \sin \theta_1 + \Delta^2}$ depends on the direction of the electron momentum \mathbf{l} with respect to the field $\theta_1 \equiv \angle(\mathbf{l}, \mathbf{B})$ – integration in the “rungs” of the Cooper ladder can be performed only over the fermionic frequency and the magnitude of the electron momenta (or, equivalently, the variable ξ_1). Consequently, the elementary building block of the ladder

$$\begin{aligned} L(\theta_1) &= T \sum_{\omega} \frac{m}{2\pi} \int d\xi_1 g_t(\omega, \mathbf{l}) g_{t'}(-\omega, -\mathbf{l}) \\ &= \frac{m}{2\pi} \ln \frac{\Lambda}{\max\{T, |t\bar{\Delta}_{\mathbf{k}_F}(\theta_1) - t'\bar{\Delta}_{-\mathbf{k}_F}(\theta_1)|\}} \end{aligned} \quad (4.38)$$

depends on θ_1 .

In principle, the dependence of L on the angle θ_1 should be taken into account when averaging over θ_1 . However, in the limit of $\Delta \ll T \ll |\alpha|k_F$, the angle-dependent term under the logarithm can be approximated as $|t\bar{\Delta}_{\mathbf{k}_F}(\theta_1) - t'\bar{\Delta}_{-\mathbf{k}_F}(\theta_1)| \approx |t - t'| |\alpha|k_F$ and, as

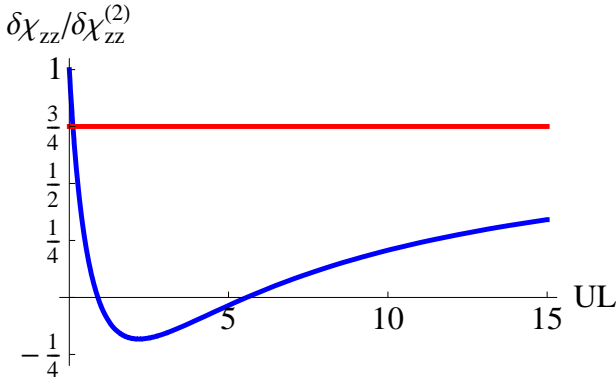


FIG. 12. Nonanalytic part of χ_{zz} , normalized by the second-order result (3.30), as a function of the Cooper-channel renormalization parameter $UL = (Um/2\pi)\ln(\Lambda/T)$. The horizontal line marks the low-temperature limit.

it was also the case for χ_{zz} , $L = (m/2\pi)\ln(\Lambda/T)$ provided that $t = t'$.

Second, the particle-hole bubbles also depend on the direction of the electron momentum, hence, the infinite-order result for the thermodynamic potential is found again by replacing the bare interaction U in the second-order diagram by $\Gamma_{s_1 s_2; s_3 s_4}(\theta_{\mathbf{k}})$ and retaining the angular dependence of the bubbles.

The RG equations for the in-plane magnetic field are considerably more complicated. The main difference is that even the RG-invariant terms depend on the magnetic field. A detailed discussion of Cooper renormalization of the scattering amplitudes and spin susceptibility for this case is given in Appendices C 2 a and C 2 b, respectively. Below, we only show the final result for the renormalized spin susceptibility

$$\delta\chi_{xx} = \frac{\chi_0}{3} \left(\frac{mU}{4\pi} \right)^2 \frac{|\alpha|k_F}{E_F} + \mathcal{O}\left(\frac{T}{\ln T}\right). \quad (4.39)$$

The T -independent term is the same as without the Cooper renormalization [cf. (3.51)]. The linear-in- T term however is suppressed by at least a factor of $1/\ln T$, similar to the case of no SOI, where it is suppressed by a $\ln^2 T$ factor.

V. SUMMARY AND DISCUSSION

We have considered a two-dimensional electron liquid in the presence of the Rashba spin-orbit interaction (SOI). The main result of this paper is that the combined effect of the electro-electron and spin-orbit interactions breaks isotropy of the spin response, whereas either of these two mechanisms does not. Namely, nonanalytic behavior of the spin susceptibility, as manifested by its temperature- and magnetic-field dependences, studied in this paper, is different for different components of the susceptibility tensor: whereas the nonanalytic behavior of

χ_{zz} is cut off at the energy scale associated with the SOI (given by $|\alpha|k_F$ for the Rashba SOI), that of χ_{xx} (and $\chi_{yy} = \chi_{xx}$) continues through the SOI energy scale. The reason for this difference is the dependence of the SOI-induced magnetic field on the electron momentum. If the external magnetic field is perpendicular to the plane of motion, its effect is simply dual to that of the SOI-field: the T dependence of χ_{zz} is cut by whichever of the two fields is larger. If the external field is in the plane of motion, it is always possible to form a virtual particle-hole pair, which mediates the long-range interaction between quasiparticles, from the states belonging to the same Rashba branch. The energy of such a pair depends on the external but not on the effective field, so that the SOI effectively drops out of the result. We have also studied a non-perturbative renormalization of the spin susceptibility in the Cooper channel of the electron-electron interaction. It turns out the RG flow of scattering amplitudes is highly non-trivial. As a result, the spin susceptibility exhibits a non-monotonic dependence on the Cooper-channel renormalization parameter ($\ln T$) and eventually saturates as a temperature-independent value, proportional to the SOI coupling $|\alpha|$.

Notably, all the results of the paper are readily applicable to the systems with large Dresselhaus SOI and negligible Rashba SOI. In this case the Rashba spin-orbit coupling should be simply replaced by the Dresselhaus spin-orbit coupling.

Now we would like to discuss possible implications of these results for (in)stability of a second-order ferromagnetic quantum critical point (QCP). This phenomenon depends crucially on the sign of the nonanalytic correction. In this regard, we should point out that we limited our analysis to the simplest possible model, which does not involve the Kohn-Luttinger superconducting instability and higher-order processes in the particle-hole channel. Therefore, the sign of our nonanalytic correction is “anomalous”, i.e., the spin susceptibility increases with the corresponding energy scale. As in the absence of the SOI, however, either of these two effects (Kohn-Luttinger and particle-hole) can reverse the sign of nonanalyticity. Therefore, it is instructive to consider consequences of both signs.

In the absence of the SOI, a nonanalyticity of the anomalous sign renders a second-order ferromagnetic QCP unstable with respect to either a first-order phase transition or a transition into a spiral state.^{3,4} This result was previously believed to be relevant only to systems with a $SU(2)$ symmetry of electron spins; in particular it was shown in Ref. 5b that there is no nonanalyticity in χ for a model case of the Ising-like exchange interaction between electrons. We have shown here that broken (by the SOI) $SU(2)$ symmetry is not sufficient for eliminating a nonanalyticity in the in-plane component of the spin susceptibility (χ_{xx}). Based on our results for the magnetic-field dependence of χ_{zz} and χ_{xx} [cf. Eqs. (3.37) and (3.53)], we can construct a model form for the free energy as a function of the magnetization M . The most inter-

esting case for us is the one in which the Zeeman energy due to spontaneous magnetization, $\sim M/m\mu_B$ is larger than the SOI energy scale $|\alpha|k_F$, so that $\chi_{xx} \neq \chi_{zz}$. Ignoring Cooper-channel renormalization, we can write the free energy as

$$F = aM^2 - b(|M_x|^3 + |M_y|^3) + M^4, \quad (5.1)$$

where $M = (M_x^2 + M_y^2 + M_z^2)^{1/2}$ and the coefficient of the quartic term was absorbed into the overall scale of F , which is irrelevant for our discussion. An important difference of this free energy, compared to the case of no SOI, is easy-plane anisotropy of the nonanalytic, cubic term. In the absence of the cubic term ($b = 0$), a second-order quantum phase transitions occurs when $a = 0$; in the paramagnetic phase, a is positive but small near the QCP. Since χ is isotropic at the mean-field level [cf. Eq. (2.8)], the regular, M^2 and M^4 terms in Eq. (5.1) are isotropic as well. If $b > 0$ (which corresponds to the anomalous sign of nonanalyticity), the cubic term leads to a minimum of F at finite M ; when the minimum value of F reaches zero, the states with zero and finite magnetization become degenerate, and a first-order phase transition occurs. The first-order critical point is specified by the following equations

$$\frac{\partial F}{\partial M_z} = 0, \quad \frac{\partial F}{\partial M_x} = 0, \quad \frac{\partial F}{\partial M_y} = 0, \quad \text{and} \quad F = 0. \quad (5.2)$$

For $a > 0$, the only root of the first equation is $M_z = 0$, i.e., there is no net magnetization in the z direction. Substituting $M_z = 0$ into the remaining equations and

employing in-plane symmetry ($M_x = M_y$), we find that the first-order phase transition occurs at $a = b^2/8$. The broken-symmetry state is an XY ferromagnet with spontaneous in-plane magnetization $M_x^{\{c\}} = M_y^{\{c\}} = b/4$. The first-order transition to an XY ferromagnet occurs if the Zeeman energy, corresponding to a jump of the magnetization at the critical point, is larger than the SOI energy, i.e., $M_x^{\{c\}}/m\mu_B \gg |\alpha|k_F$. In the opposite case, the SOI is irrelevant, and the first-order transition is to a Heisenberg ferromagnet.

If the nonanalyticity is of the “normal” sign ($b < 0$), the transition remains second order and occurs at $a = 0$. However, the critical indices are different for the in-plane and transverse magnetization: in the broken-symmetry phase ($a < 0$), $M_x = M_y \propto (-a)$ while $M_z \propto (-a)^{1/2}$. Since $|a| \ll 1$, the resulting state is an Ising-like ferromagnet with $M_z \gg M_x = M_y$.

A detailed study of the $|\mathbf{q}|$ dependence of χ in the presence of the SOI will be presented elsewhere.²⁴

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Appendix A: Temperature dependence of the spin susceptibility of free Rashba fermions

In this Appendix, we consider the temperature dependence of the spin susceptibility of free 2D electrons in the presence of the Rashba SOI. The transverse and parallel spin susceptibilities, $\chi_{zz}^{\{0\}}$ and $\chi_{xx}^{\{0\}}$, are given by

$$\chi_{zz}^{\{0\}} = - \sum_K \text{Tr} \left[\hat{G}(K) \hat{\sigma}_z \hat{G}(K') \hat{\sigma}_z \right] \quad (A1)$$

$$\chi_{xx}^{\{0\}} = - \sum_K \text{Tr} \left[\hat{G}(K) \hat{\sigma}_x \hat{G}(K') \hat{\sigma}_x \right], \quad (A2)$$

where $K = (\omega, \mathbf{k})$ and $K' = (\omega, \mathbf{k} + \mathbf{q})$ with $q \rightarrow 0$. Evaluating the traces, we obtain for $\chi_{zz}^{\{0\}}$

$$\chi_{zz}^{\{0\}} = -T \sum_{\omega} \int \frac{d^2k}{(2\pi)^2} \sum_{s,t} \frac{1}{2} (1 - st) g_s(\omega, \mathbf{k}) g_t(\omega, \mathbf{k}) = -2T \sum_{\omega} \int \frac{dkk}{2\pi} g_+(\omega, \mathbf{k}) g_-(\omega, \mathbf{k}), \quad (A3)$$

where we took advantage of the isotropy of $g_{\pm}(\omega, \mathbf{k})$ and put $q = 0$, because the poles in the Green's functions of different branches reside on opposite sides of the real axis. We see that $\chi_{zz}^{\{0\}}$ is determined only by inter-subband transitions. Similarly, we obtain for $\chi_{xx}^{\{0\}}$

$$\chi_{xx}^{\{0\}} = -T \sum_{\omega} \int \frac{d^2k}{(2\pi)^2} \sum_{s,t} \frac{1}{2} [1 - st \cos(2\theta_{\mathbf{k}})] g_s(\omega, \mathbf{k}) g_t(\omega, \mathbf{k} + \mathbf{q})|_{q \rightarrow 0}. \quad (A4)$$

Since $\cos \theta_{\mathbf{k}}$ averages to zero, $\chi_{xx}^{\{0\}}$ can be written as

$$\chi_{xx}^{\{0\}} = \frac{1}{2}\chi_{zz}^{\{0\}} + \frac{1}{2}\delta\chi_{xx}^{\{0\}}, \quad (\text{A5})$$

where

$$\delta\chi_{xx}^{\{0\}} \equiv -T \sum_{\omega} \int \frac{d^2k}{(2\pi)^2} [g_+(\omega, \mathbf{k})g_+(\omega, \mathbf{k} + \mathbf{q}) + g_-(\omega, \mathbf{k})g_-(\omega, \mathbf{k} + \mathbf{q})] |_{q \rightarrow 0} \quad (\text{A6})$$

is the contribution from intra-subband transitions, absent in $\chi_{zz}^{\{0\}}$.

Let us evaluate $\chi_{zz}^{\{0\}}$ first. Performing the fermionic Matsubara sum, we obtain

$$\chi_{zz}^{\{0\}} = -2 \int_0^\infty \frac{kdk}{2\pi} T \sum_{\omega} \frac{1}{2\alpha k} \left(\frac{1}{i\omega - \xi_{\mathbf{k}} - \alpha k} - \frac{1}{i\omega - \xi_{\mathbf{k}} + \alpha k} \right) = \frac{1}{2\pi\alpha} \int_0^\infty dk [n_F(\xi_{\mathbf{k}} - \alpha k) - n_F(\xi_{\mathbf{k}} + \alpha k)] \quad (\text{A7})$$

with a Fermi function $n_F(\epsilon) = [e^{(\epsilon - \mu)/T} + 1]^{-1}$ and $\xi_{\mathbf{k}} = k^2/2m - \mu$. Changing variables in the first integral to $\xi_{\mathbf{k}} - \alpha k = \epsilon_- - \mu$, we find two roots: $k_-^{(1)} = m\alpha - \sqrt{(m\alpha)^2 + 2m\epsilon_-}$, valid for $-\epsilon_0 < \epsilon_- < 0$ with $dk_-^{(1)}/d\epsilon_- < 0$, and $k_-^{(2)} = m\alpha + \sqrt{(m\alpha)^2 + 2m\epsilon_-}$, valid for $\epsilon_- > 0$ with $dk_-^{(2)}/d\epsilon_- > 0$, where $\epsilon_0 \equiv m\alpha^2/2$. Similarly, we change variables in the second integral to $\xi_{\mathbf{k}} + \alpha k = \epsilon_+ - \mu$ and obtain only one positive root $k_+ = -m\alpha + \sqrt{(m\alpha)^2 + 2m\epsilon_+}$, valid for $\epsilon_+ > 0$ with $dk_+/d\epsilon_+ > 0$. Notice that the absolute values of the (inverse) group velocities are the same for both branches: $|dk_-^{(1,2)}/d\epsilon_-| = |dk_+/d\epsilon_+| = m[(m\alpha)^2 + 2m\epsilon_{\pm}]^{-1/2}$. Therefore,

$$\chi_{zz}^{\{0\}} = \frac{1}{2\pi\alpha} \left(\int_{-\epsilon_0}^0 d\epsilon \left| \frac{dk_-^{(1)}}{d\epsilon} \right| + \int_0^\infty d\epsilon \left| \frac{dk_-^{(2)}}{d\epsilon} \right| - \int_0^\infty d\epsilon \left| \frac{dk_+}{d\epsilon} \right| \right) n_F(\epsilon) = \frac{m}{2\pi\alpha} \int_{-\epsilon_0}^0 d\epsilon \frac{n_F(\epsilon)}{\sqrt{(m\alpha)^2 + 2m\epsilon}}, \quad (\text{A8})$$

where we dropped the index on the integration variable ϵ . Notably, the high energy contributions from the two Rashba branches cancel each other for any value of the chemical potential and the spin susceptibility is determined exclusively by the bottom part of the lower Rashba branch. Integration by parts yields

$$\chi_{zz}^{\{0\}} = \chi_0 \left(n_F(0) - \int_{-\epsilon_0}^0 d\epsilon \sqrt{1 + \frac{\epsilon}{\epsilon_0}} \frac{\partial n_F(\epsilon)}{\partial \epsilon} \right), \quad (\text{A9})$$

where $n_F(0) = [e^{-\mu/T} + 1]^{-1}$. In order to evaluate this integral, it is convenient to consider three limiting cases.

For $T \ll \epsilon_0 \ll \mu$, i.e., when both Rashba subbands are occupied and the temperature is lower than the minimum of the lower subband, we approximate $n_F(0) \approx 1 - e^{-\mu/T}$, $-T\partial n_F(\epsilon)/\partial \epsilon \approx e^{(\epsilon - \mu)/T}$, and $\sqrt{1 + \epsilon/\epsilon_0} \approx 1 + \epsilon/2\epsilon_0$, so that

$$\chi_{zz}^{\{0\}} = \chi_0 \left[(1 - e^{-\mu/T}) + \frac{1}{T} e^{-\mu/T} \int_0^\infty d\epsilon \left(1 - \frac{\epsilon}{2\epsilon_0} \right) e^{-\epsilon/T} \right] = \chi_0 \left(1 - \frac{T}{2\epsilon_0} e^{-\mu/T} \right). \quad (\text{A10})$$

At $T = 0$, $\chi_{zz}^{\{0\}} = \chi_0$.

For $\epsilon_0 \ll T \ll \mu$, i.e., when again both Rashba subbands are occupied but the temperature is higher than the minimum of the lower subband, we keep the same approximations for $n_F(0)$ and $-T\partial n_F(\epsilon)/\partial \epsilon$ but neglect the ϵ dependence of the Fermi function in the integrand:

$$- \int_{-\epsilon_0}^{\{0\}} d\epsilon \sqrt{1 + \frac{\epsilon}{\epsilon_0}} \frac{\partial n_F(\epsilon)}{\partial \epsilon} \approx \frac{1}{T} e^{-\mu/T} \int_0^{\epsilon_0} d\epsilon \sqrt{1 + \frac{\epsilon}{\epsilon_0}} e^{-\epsilon/T} \approx \frac{1}{T} e^{-\mu/T} \int_0^{\epsilon_0} d\epsilon \sqrt{1 + \frac{\epsilon}{\epsilon_0}} = \frac{2\epsilon_0}{3T} e^{-\mu/T}. \quad (\text{A11})$$

Thus,

$$\chi_{zz}^{\{0\}} = \chi_0 \left[1 - \left(1 - \frac{2\epsilon_0}{3T} \right) e^{-\mu/T} \right]. \quad (\text{A12})$$

Finally, for $\mu < 0$, i.e., when only the lower Rashba subband is occupied, the first term in Eq. (A9) gives only an exponentially weak temperature dependence, while the Sommerfeld expansion of the second term generates a T^2 contribution because the density of states depends on ϵ :

$$\chi_{zz}^{\{0\}} = \chi_0 \left[\sqrt{1 - |\mu|/\epsilon_0} - \frac{\pi^2}{24} \left(\frac{T}{\epsilon_0} \right)^2 \frac{1}{(1 - |\mu|/\epsilon_0)^{3/2}} \right] \quad (\text{A13})$$

At zero temperature, $\chi_{zz}^{\{0\}} = \chi_0 \sqrt{1 - |\mu|/\epsilon_0}$ vanishes at the bottom of the lower subband.

We now calculate $\delta\chi_{xx}^{\{0\}}$ given by Eq. (A6), which can be written as

$$\delta\chi_{xx}^{\{0\}} = - \int_0^\infty \frac{dk}{2\pi} \left(\frac{\partial n_F(\epsilon_-)}{\partial \epsilon_-} + \frac{\partial n_F(\epsilon_+)}{\partial \epsilon_+} \right). \quad (\text{A14})$$

The change of variables is straightforward in the second integral, since the equation $\epsilon_+ - \mu = \xi_{\mathbf{k}} + \alpha k$ has only one positive root k_+ and the density of states is given by

$$\nu_+(\epsilon) = \frac{k_+}{2\pi} \frac{dk_+}{d\epsilon} = \frac{m}{2\pi} \frac{\sqrt{1 + \epsilon/\epsilon_0} - 1}{\sqrt{1 + \epsilon/\epsilon_0}} \quad (\text{A15})$$

with $\epsilon_0 \equiv m\alpha^2/2$. In the first integral more care must be taken: for $\epsilon > 0$, we have $k_-^{(2)} = m\alpha + \sqrt{(m\alpha)^2 + 2m\epsilon}$ and the density of states is given by

$$\nu_-^{>}(\epsilon) = \frac{k_-^{(2)}}{2\pi} \frac{dk_-^{(2)}}{d\epsilon} = \frac{m}{2\pi} \frac{\sqrt{1 + \epsilon/\epsilon_0} + 1}{\sqrt{1 + \epsilon/\epsilon_0}}; \quad (\text{A16})$$

however, for $-\epsilon_0 < \epsilon < 0$ both roots enter the density of states

$$\nu_-^{<}(\epsilon) = \int_0^\infty \frac{k dk}{2\pi} \delta(\epsilon - \epsilon_-) = \frac{1}{2\pi} \left(k_-^{(1)} \left| \frac{dk_-^{(1)}}{d\epsilon_-} \right| + k_-^{(2)} \left| \frac{dk_-^{(2)}}{d\epsilon_-} \right| \right) = \frac{m}{\pi} \frac{1}{\sqrt{1 + \epsilon/\epsilon_0}}. \quad (\text{A17})$$

Summing up the contributions from all energies, we find

$$\delta\chi_{xx}^{\{0\}} = \left(\int_{-\epsilon_0}^0 d\epsilon \nu_-^{<}(\epsilon) + \int_0^\infty d\epsilon \nu_-^{>}(\epsilon) + \int_0^\infty d\epsilon \nu_+(\epsilon) \right) \left(-\frac{\partial n_F(\epsilon)}{\partial \epsilon} \right) = \int_{-\epsilon_0}^0 d\epsilon \nu_-^{<}(\epsilon) \left(-\frac{\partial n_F(\epsilon)}{\partial \epsilon} \right) + \chi_0 n_F(0), \quad (\text{A18})$$

where the second and third integrals are easily evaluated because ϵ drops out of the sum $\nu_+ + \nu_-^{>} = m/\pi$. Combining the above result with Eqs. (A2) and (A8), we get

$$\chi_{xx}^{\{0\}} = \chi_0 \left(n_F(0) - \int_{-\epsilon_0}^0 d\epsilon \frac{1 + \epsilon/2\epsilon_0}{\sqrt{1 + \epsilon/\epsilon_0}} \frac{\partial n_F(\epsilon)}{\partial \epsilon} \right). \quad (\text{A19})$$

For $T \ll \epsilon_0 \ll \mu$, we approximate $n_F(0) \approx 1 - e^{-\mu/T}$ and $-T \partial n_F(\epsilon)/\partial \epsilon = e^{(\epsilon-\mu)/T} (e^{(\epsilon-\mu)/T} + 1)^{-2} \approx e^{(\epsilon-\mu)/T}$ as before, and expand $(1 + \epsilon/2\epsilon_0)/\sqrt{1 + \epsilon/\epsilon_0} \approx 1 + \epsilon^2/8\epsilon_0^2$, so that

$$\chi_{xx}^{\{0\}} = \chi_0 \left[n_F(0) + \frac{1}{T} e^{-\mu/T} \int_0^\infty d\epsilon \left(1 + \frac{\epsilon^2}{8\epsilon_0^2} \right) e^{-\epsilon/T} \right] = \chi_0 \left(1 + \frac{T^2}{4\epsilon_0^2} e^{-\mu/T} \right). \quad (\text{A20})$$

For $\epsilon_0 \ll T \ll \mu$, we keep the same approximations for the Fermi function but, as it was also the case for $\chi_{zz}^{\{0\}}$, neglect the ϵ dependence of the Fermi function in the integrand

$$- \int_{-\epsilon_0}^0 d\epsilon \frac{1 + \epsilon/2\epsilon_0}{\sqrt{1 + \epsilon/\epsilon_0}} \frac{\partial n_F(\epsilon)}{\partial \epsilon} \approx \frac{1}{T} e^{-\mu/T} \int_0^{\epsilon_0} d\epsilon \frac{1 - \epsilon/2\epsilon_0}{\sqrt{1 - \epsilon/\epsilon_0}} e^{-\epsilon/T} \approx \frac{1}{T} e^{-\mu/T} \int_0^{\epsilon_0} d\epsilon \frac{1 - \epsilon/2\epsilon_0}{\sqrt{1 - \epsilon/\epsilon_0}} = \frac{4\epsilon_0}{3T} e^{-\mu/T}. \quad (\text{A21})$$

Thus,

$$\chi_{xx}^{\{0\}} = \chi_0 \left[1 - \left(1 - \frac{4\epsilon_0}{3T} \right) e^{-\mu/T} \right]. \quad (\text{A22})$$

Finally, for $\mu < 0$, the Sommerfeld expansion of the second term in Eq. (A19) yields

$$\chi_{xx}^{\{0\}} = \chi_0 \left[\frac{1 - |\mu|/2\epsilon_0}{(1 - |\mu|/\epsilon_0)^{1/2}} + \frac{\pi^2}{48} \left(\frac{T}{\epsilon_0} \right)^2 \frac{2 + |\mu|/\epsilon_0}{(1 - |\mu|/\epsilon_0)^{5/2}} \right] \quad (\text{A23})$$

for $T \ll \min\{|\mu|, \epsilon_0 - |\mu|\}$. At zero temperature, $\chi_{xx}^{\{0\}} = \chi_0 (1 - |\mu|/2\epsilon_0) / (1 - |\mu|/\epsilon_0)^{1/2}$ diverges at the bottom of the lower subband.

Appendix B: Absence of a small- q singularity in the static polarization bubble with spin-orbit interaction

In Secs. III and III D, we argued that there is no contribution to the nonanalytic behavior of the spin susceptibility from the region of small bosonic momenta, $q \ll k_F$. This statement contradicts Ref. 32 where it was argued that, in the presence of the SOI, a static particle-hole bubble has a square-root singularity at $q = q_0 \equiv 2m\alpha$ (in addition to the Kohn anomaly which is also modified by the SOI). For a weak SOI, q_0 is much smaller than k_F and thus the region of small q may also contribute to the nonanalytic behavior. Later on, however, Refs. 28 and 33 showed that there is no singularity at $q = q_0$. According to Ref. 28, the reason is related to a subtlety in approaching the static limit of a dynamic bubble. While we agree with the authors of Refs. 28 in that there are no small- q singularities in the bubble, we find that a cancellation of singular terms occurs in the calculation of a purely static bubble. The same result was obtained in an unpublished work.³⁴ For the sake of completeness, we present our derivation in this Appendix.

Evaluating the spin trace, we obtain for the static polarization bubble

$$\Pi(q) \equiv \sum_K \text{Tr}[\hat{G}_{\omega, \mathbf{k}+\mathbf{q}} \hat{G}_{\omega, \mathbf{k}}] = \frac{1}{2} \sum_K \sum_{s,t} [1 + st \cos(\varphi_{\mathbf{k}+\mathbf{q}} - \varphi_{\mathbf{k}})] g_s(\omega, \mathbf{k} + \mathbf{q}) g_t(\omega, \mathbf{k}), \quad (\text{B1})$$

where, as before, $K = (\omega, \mathbf{k})$, $\varphi_{\mathbf{k}+\mathbf{q}} \equiv \angle(\mathbf{k} + \mathbf{q}, \mathbf{e}_x)$ and $\varphi_{\mathbf{k}} \equiv \angle(\mathbf{k}, \mathbf{e}_x)$. We divide $\Pi(q)$ into intra- and intersubband contributions as

$$\Pi(q) = \Pi_{++}(q) + \Pi_{--}(q) + \Pi_{\pm}(q), \quad (\text{B2})$$

where

$$\Pi_{\pm\pm}(q) \equiv \frac{1}{2} \sum_K [1 + \cos(\varphi_{\mathbf{k}+\mathbf{q}} - \varphi_{\mathbf{k}})] g_{\pm}(\omega, \mathbf{k} + \mathbf{q}) g_{\pm}(\omega, \mathbf{k}), \quad (\text{B3a})$$

$$\begin{aligned} \Pi_{\pm}(q) &\equiv \frac{1}{2} \sum_K [1 - \cos(\varphi_{\mathbf{k}+\mathbf{q}} - \varphi_{\mathbf{k}})] [g_{+}(\omega, \mathbf{k} + \mathbf{q}) g_{-}(\omega, \mathbf{k}) + g_{-}(\omega, \mathbf{k} + \mathbf{q}) g_{+}(\omega, \mathbf{k})] \\ &= \sum_K [1 - \cos(\varphi_{\mathbf{k}+\mathbf{q}} - \varphi_{\mathbf{k}})] g_{+}(\omega, \mathbf{k} + \mathbf{q}) g_{-}(\omega, \mathbf{k}). \end{aligned} \quad (\text{B3b})$$

In the last line, we employed obvious symmetries of the Green's function.

First, we focus on the intersubband part, $\Pi_{\pm}(q)$. Summation over the Matsubara frequency yields

$$T \sum_{\omega} g_{+}(\mathbf{p}, \omega) g_{-}(\mathbf{k}, \omega) = \frac{n_F(\xi_{\mathbf{p}}^{+}) - n_F(\xi_{\mathbf{k}}^{-})}{\xi_{\mathbf{p}}^{+} - \xi_{\mathbf{k}}^{-}}, \quad (\text{B4})$$

where $\mathbf{p} = \mathbf{k} + \mathbf{q}$, $\xi_{\mathbf{k}}^{\pm} = \xi_{\mathbf{k}} \pm \alpha k$ and, as before, $\xi_{\mathbf{k}} = k^2/2m - E_F$. Introducing additional integration over the momentum \mathbf{p} , as it was done in Ref. 32, Eq. (B3b) can be re-written as

$$\Pi_{\pm}(q) = \frac{2}{(2\pi)^2} \int_0^{2\pi} d\theta \int_0^{\infty} dk k \int_0^{\infty} dp p \delta(p^2 - |\mathbf{k} + \mathbf{q}|^2) \left(1 - \frac{\mathbf{k} \cdot (\mathbf{k} + \mathbf{q})}{kp}\right) \frac{n_F(\xi_{\mathbf{p}}^{+}) - n_F(\xi_{\mathbf{k}}^{-})}{\xi_{\mathbf{p}}^{+} - \xi_{\mathbf{k}}^{-}}, \quad (\text{B5})$$

where $\theta = \angle(\mathbf{k}, \mathbf{q})$. Integration over θ yields

$$\int_0^{2\pi} d\theta \delta(p^2 - k^2 - q^2 - 2kq \cos \theta) \left(1 - \frac{k^2 + kq \cos \theta}{kp}\right) = \frac{1}{kp} \frac{q^2 - (k-p)^2}{\sqrt{(k+q)^2 - p^2} \sqrt{p^2 - (k-q)^2}}, \quad (\text{B6})$$

which imposes a constraint on the range of integration over p , i.e., $|k - q| < p < k + q$. Since we assume that $q \ll k \approx p \approx k_F$, Eq. (B6) can be simplified to

$$\frac{\sqrt{q^2 - (k-p)^2}}{2k_F^3}, \quad (\text{B7})$$

and $\Pi_{\pm}(q)$ becomes

$$\Pi_{\pm}(q) = \frac{1}{4\pi^2 k_F^3} \int_0^{\infty} dk k \int_{|k-q|}^{k+q} dp p \sqrt{q^2 - (k-p)^2} \frac{n_F(\xi_{\mathbf{p}}^{+}) - n_F(\xi_{\mathbf{k}}^{-})}{\xi_{\mathbf{p}}^{+} - \xi_{\mathbf{k}}^{-}}. \quad (\text{B8})$$

For a weak SOI ($m|\alpha| \ll k_F$), $\xi_{\mathbf{k}}^{\pm} \approx \xi_{\mathbf{k}} \pm \alpha k_F$. Switching from integration over k and p to integration over $\xi_{\mathbf{k}}$ and $\xi_{\mathbf{p}}$, we find

$$\Pi_{\pm}(q) = \frac{1}{4\pi^2 v_F^3 k_F} \int_{-\infty}^{\infty} d\xi_{\mathbf{k}} \int_{\xi_{\mathbf{k}} - v_F q}^{\xi_{\mathbf{k}} + v_F q} d\xi_{\mathbf{p}} \sqrt{(v_F q)^2 - (\xi_{\mathbf{k}} - \xi_{\mathbf{p}})^2} \frac{n_F(\xi_{\mathbf{p}} + \alpha k_F) - n_F(\xi_{\mathbf{k}} - \alpha k_F)}{\xi_{\mathbf{p}} - \xi_{\mathbf{k}} + 2\alpha k_F}. \quad (\text{B9})$$

Shifting the integration variables as $\xi_{\mathbf{p}} \rightarrow \xi_{\mathbf{p}} - \alpha k_F$, $\xi_{\mathbf{k}} \rightarrow \xi_{\mathbf{k}} + \alpha k_F$, we eliminate the dependence of the Fermi functions on αk_F . Assuming also that $T = 0$, we obtain

$$\Pi_{\pm}(q) = \frac{1}{4\pi^2 v_F^3 k_F} \int_{-\infty}^{\infty} d\xi_{\mathbf{k}} \int_{\xi_{\mathbf{k}} + 2\alpha k_F - v_F q}^{\xi_{\mathbf{k}} + 2\alpha k_F + v_F q} d\xi_{\mathbf{p}} \sqrt{(v_F q)^2 - (\xi_{\mathbf{p}} - \xi_{\mathbf{k}} - 2\alpha k_F)^2} \frac{\Theta(-\xi_{\mathbf{p}}) - \Theta(-\xi_{\mathbf{k}})}{\xi_{\mathbf{p}} - \xi_{\mathbf{k}}}, \quad (\text{B10})$$

where $\Theta(x)$ is the step function. Notice that the integrand is finite only if $\xi_{\mathbf{k}} \xi_{\mathbf{p}} < 0$, which imposes further constraints on the integration range.

We will now prove that $\Pi_{\pm}(q)$ given by Eq. (B10) is continuous at $q = q_0$. To this end, it is convenient to consider the cases of $q < q_0$ and $q > q_0$. Combining all the constraints together, we find that Π_{\pm} for $q < q_0$ can be written as

$$\begin{aligned} \Pi_{\pm}^<(q) \equiv \Pi_{\pm}(q < q_0) = & -\frac{1}{4\pi^2 v_F^3 k_F} \left(\int_{-v_F q - 2\alpha k_F}^{v_F q - 2\alpha k_F} d\xi_{\mathbf{k}} \int_0^{\xi_{\mathbf{k}} + 2\alpha k_F + v_F q} d\xi_{\mathbf{p}} \right. \\ & \left. + \int_{v_F q - 2\alpha k_F}^0 d\xi_{\mathbf{k}} \int_{\xi_{\mathbf{k}} + 2\alpha k_F - v_F q}^{\xi_{\mathbf{k}} + 2\alpha k_F + v_F q} d\xi_{\mathbf{p}} \right) \frac{\sqrt{(v_F q)^2 - (\xi_{\mathbf{p}} - \xi_{\mathbf{k}} - 2\alpha k_F)^2}}{\xi_{\mathbf{p}} - \xi_{\mathbf{k}}}, \end{aligned} \quad (\text{B11})$$

Reversing the sign of $\xi_{\mathbf{k}}$, absorbing $\xi_{\mathbf{k}}$ into $\xi_{\mathbf{p}}$, and defining the dimensionless variables $x = \xi_{\mathbf{k}}/v_F q$ and $y = \xi_{\mathbf{p}}/v_F q$, we obtain

$$\Pi_{\pm}^<(q) = -\frac{q^2}{4\pi^2 v_F k_F} \left(\int_{\beta-1}^{\beta+1} dx \int_x^{\beta+1} dy + \int_0^{\beta-1} dx \int_{\beta-1}^{\beta+1} dy \right) \frac{\sqrt{1 - (y - \beta)^2}}{y}, \quad (\text{B12})$$

where $\beta \equiv q_0/q > 1$. Next, we switch the order of integration in the first term, so that the integrals over x can be readily evaluated

$$\begin{aligned} \Pi_{\pm}^<(q) = & -\frac{q^2}{4\pi^2 v_F k_F} \int_{\beta-1}^{\beta+1} dy \left(\int_{\beta-1}^y dx + \int_0^{\beta-1} dx \right) \frac{\sqrt{1 - (y - \beta)^2}}{y} \\ = & -\frac{q^2}{4\pi^2 v_F k_F} \int_{\beta-1}^{\beta+1} dy \sqrt{1 - (y - \beta)^2} = -\frac{m}{2\pi} \left(\frac{q}{2k_F} \right)^2. \end{aligned} \quad (\text{B13})$$

For $q > q_0$, we have

$$\begin{aligned} \Pi_{\pm}^>(q) \equiv \Pi_{\pm}(q > q_0) = & \frac{1}{4\pi^2 v_F^3 k_F} \left(\int_0^{v_F q - 2\alpha k_F} d\xi_{\mathbf{k}} \int_{\xi_{\mathbf{k}} + 2\alpha k_F - v_F q}^0 d\xi_{\mathbf{p}} \right. \\ & \left. - \int_{-v_F q - 2\alpha k_F}^0 d\xi_{\mathbf{k}} \int_0^{\xi_{\mathbf{k}} + 2\alpha k_F + v_F q} d\xi_{\mathbf{p}} \right) \frac{\sqrt{(v_F q)^2 - (\xi_{\mathbf{p}} - \xi_{\mathbf{k}} - 2\alpha k_F)^2}}{\xi_{\mathbf{p}} - \xi_{\mathbf{k}}}. \end{aligned} \quad (\text{B14})$$

Manipulations similar to those for the previous case yield

$$\Pi_{\pm}^>(q) = -\frac{q^2}{4\pi^2 v_F k_F} \left(\int_0^{1-\beta} dx \int_x^{1-\beta} dy \frac{\sqrt{1 - (y + \beta)^2}}{y} + \int_0^{1+\beta} dx \int_x^{1+\beta} dy \frac{\sqrt{1 - (y - \beta)^2}}{y} \right) \quad (\text{B15})$$

with $\beta = q_0/q < 1$. Interchanging the order of integrations over x and y , we find

$$\Pi_{\pm}(q > 2m\alpha) = -\frac{q^2}{4\pi^2 v_F k_F} \left(\int_0^{1-\beta} dy \sqrt{1 - (y + \beta)^2} + \int_0^{1+\beta} dy \sqrt{1 - (y - \beta)^2} \right) = -\frac{m}{2\pi} \left(\frac{q}{2k_F} \right)^2. \quad (\text{B16})$$

Since $\Pi_{\pm}^<(q = q_0 - 0^+) = \Pi_{\pm}^>(q = q_0 + 0^+)$, the function $\Pi_{\pm}(q) = -(m/2\pi)(q/2k_F)^2$ is continuous at $q = q_0$ and, thus, there is no singularity in the static particle-hole response function. In addition, $\Pi_{\pm}(q)$ does not depend on the SOI. However, since there is no q^2 term in the 2D bubble for $q \leq 2k_F$ in the absence of the SOI, the q^2 term must be canceled out by similar terms in $\Pi_{++}(q)$ and $\Pi_{--}(q)$, which is what we will show below.

Having proven that $\Pi_{\pm}(q)$ is an analytic function of q , we can re-derive its q dependence simply by expanding the combination $\varphi_{\mathbf{k}+\mathbf{q}} - \varphi_{\mathbf{k}}$ in Eq. (B3b) for $q \ll k_F$ as $\varphi_{\mathbf{k}+\mathbf{q}} - \varphi_{\mathbf{k}} \approx (q/k_F) \sin \varphi_{\mathbf{k}\mathbf{q}}$, where $\varphi_{\mathbf{k}\mathbf{q}} \equiv \angle(\mathbf{k}, \mathbf{q})$; then $1 - \cos(\varphi_{\mathbf{k}+\mathbf{q}} - \varphi_{\mathbf{k}}) \approx (q/k_F)^2 \sin^2(\varphi_{\mathbf{k}\mathbf{q}})/2$. Since we already have a factor of q^2 up front, the Green's functions in Eq. (B3b) can be evaluated at $q = 0$. Accordingly, Eq. (B3b) becomes

$$\begin{aligned} \Pi_{\pm}(q) &= \frac{1}{2} \left(\frac{q}{k_F} \right)^2 \sum_K \sin^2 \varphi_{\mathbf{k}\mathbf{q}} g_{+}(\mathbf{k}, \omega) g_{-}(\mathbf{k}, \omega) = \\ &= \frac{m}{2\pi} \left(\frac{q}{2k_F} \right)^2 \int d\xi_{\mathbf{k}} \frac{n_F(\xi_{\mathbf{k}}^{+}) - n_F(\xi_{\mathbf{k}}^{-})}{\xi_{\mathbf{k}}^{+} - \xi_{\mathbf{k}}^{-}} = -\frac{m}{2\pi} \left(\frac{q}{2k_F} \right)^2 \int_{-\alpha k_F}^{\alpha k_F} d\xi_{\mathbf{k}} \frac{1}{2\alpha k_F} = -\frac{m}{2\pi} \left(\frac{q}{2k_F} \right)^2. \end{aligned} \quad (\text{B17})$$

Expanding Eq. (B3a) for the intraband contribution to the bubble also to second order in q , we obtain

$$\Pi_{\pm\pm}(Q) = \sum_K g_{\pm}(\mathbf{k} + \mathbf{q}, \omega) g_{\pm}(\mathbf{k}, \omega)|_{\mathbf{q} \rightarrow 0} - \frac{1}{4} \left(\frac{q}{k_F} \right)^2 \sum_K \sin^2 \varphi_{\mathbf{k}\mathbf{q}} g_{\pm}(\mathbf{k} + \mathbf{q}, \omega) g_{\pm}(\mathbf{k}, \omega)|_{q \rightarrow 0} = -\frac{m}{2\pi} + \frac{m}{4\pi} \left(\frac{q}{2k_F} \right)^2, \quad (\text{B18})$$

since

$$\sum_K g_{\pm}(\mathbf{k} + \mathbf{q}, \omega) g_{\pm}(\mathbf{k}, \omega)|_{\mathbf{q} \rightarrow 0} = -m \int \frac{d\xi_{\mathbf{k}}}{2\pi} \frac{\Theta(\xi_{\mathbf{k}+\mathbf{q}}^{\pm}) - \Theta(\xi_{\mathbf{k}}^{\pm})}{\xi_{\mathbf{k}+\mathbf{q}}^{\pm} - \xi_{\mathbf{k}}^{\pm}} \Big|_{q \rightarrow 0} = -m \int \frac{d\xi_{\mathbf{k}}}{2\pi} \delta(\xi_{\mathbf{k}} \pm |\alpha|k_F) = -\frac{m}{2\pi}. \quad (\text{B19})$$

Thereby the total bubble (B2)

$$\Pi(q) = -\frac{m}{\pi} \quad (\text{B20})$$

is independent of q for $q \ll 2k_F$.

Appendix C: Renormalization of the scattering amplitudes in the Cooper channel in the presence of the magnetic field

In this Appendix, we present the derivation of the RG flow equations for the scattering amplitudes in the Cooper channel in the presence of the magnetic field. These amplitudes are then used to find the non-perturbative results for the spin susceptibility. Since the amplitudes are renormalized quite differently for the field applied perpendicularly and parallel to the 2DEG plane, we will treat these two cases separately.

1. Transverse magnetic field

a. RG flow of the scattering amplitudes

If the magnetic field is transverse to the 2DEG plane, $\mathbf{B} = B\mathbf{e}_z$, the eigenvectors of the Hamiltonian (1.1) read

$$|\mathbf{k}, s\rangle = \frac{1}{\sqrt{N_s(k)}} \begin{pmatrix} (\Delta - s\bar{\Delta}_{\mathbf{k}})ie^{-i\theta_{\mathbf{k}}}/\alpha k \\ 1 \end{pmatrix}, \quad (\text{C1})$$

where $N_s(k) = 2 + 2\Delta(\Delta - s\bar{\Delta}_{\mathbf{k}})/(\alpha k)^2$ is the normalization factor and, as before, $\bar{\Delta}_{\mathbf{k}} \equiv (\Delta^2 + \alpha^2 k^2)^{-1/2}$ is the effective Zeeman energy. Since, by assumption, $|\alpha|k_F \ll E_F$, we approximate $\bar{\Delta}_{\mathbf{k}}$ by $\bar{\Delta}_{\mathbf{k}_F}$. Substituting the above eigenvectors into Eq. (4.17), we find the scattering amplitude

$$\begin{aligned} \Gamma_{s_1 s_2; s_4 s_3}^{(1)}(\mathbf{k}, \mathbf{k}'; \mathbf{p}, \mathbf{p}') &= U \left(\prod_{i=1}^4 \frac{1}{\sqrt{2 + \frac{2\Delta(\Delta - s_i \bar{\Delta}_{\mathbf{k}_F})}{\alpha^2 k_F^2}}} \right) \left[1 + \frac{1}{\alpha^2 k_F^2} (\Delta - s_1 \bar{\Delta}_{\mathbf{k}_F})(\Delta - s_3 \bar{\Delta}_{\mathbf{k}_F}) e^{i(\theta_{\mathbf{p}} - \theta_{\mathbf{k}})} \right] \\ &\quad \times \left[1 + \frac{1}{\alpha^2 k_F^2} (\Delta - s_2 \bar{\Delta}_{\mathbf{k}_F})(\Delta - s_4 \bar{\Delta}_{\mathbf{k}_F}) e^{i(\theta_{\mathbf{p}'} - \theta_{\mathbf{k}'})} \right]. \end{aligned} \quad (\text{C2})$$

To find the spin susceptibility, we need to know the scattering amplitude to second order in the magnetic field. Expanding Eq. (C2) to second order in $\delta \equiv \Delta/|\alpha|k_F$ (note that $s_i^2 = 1$ and $s_i^3 = s_i$) and projecting the amplitude onto the Cooper channel, where the momenta are correlated in such a way that $\mathbf{k}' = -\mathbf{k}$ and $\mathbf{p}' = -\mathbf{p}$ or, equivalently, $\theta_{\mathbf{k}'} = \theta_{\mathbf{k}} + \pi$ and $\theta_{\mathbf{p}'} = \theta_{\mathbf{p}} + \pi$, we obtain

$$\Gamma_{s_1 s_2; s_3 s_4}^{(1)}(\mathbf{k}, -\mathbf{k}; \mathbf{p}, -\mathbf{p}) = U_{s_1 s_2; s_3 s_4} + V_{s_1 s_2; s_3 s_4} e^{i(\theta_{\mathbf{p}} - \theta_{\mathbf{k}})} + W_{s_1 s_2; s_3 s_4} e^{2i(\theta_{\mathbf{p}} - \theta_{\mathbf{k}})}, \quad (\text{C3})$$

where we introduced partial amplitudes

$$U_{s_1 s_2; s_3 s_4} = \frac{U}{4} + \frac{U}{8}(s_1 + s_2 + s_3 + s_4)\delta + \frac{U}{16}(s_1 s_2 + s_1 s_3 + s_1 s_4 + s_2 s_3 + s_2 s_4 + s_3 s_4 - 2)\delta^2 + \mathcal{O}(\delta^3), \quad (\text{C4})$$

$$V_{s_1 s_2; s_3 s_4} = \frac{U}{4}(s_1 s_3 + s_2 s_4) + \frac{U}{8}(s_1 s_2 s_3 + s_1 s_2 s_4 + s_1 s_3 s_4 + s_2 s_3 s_4 - s_1 - s_2 - s_3 - s_4)\delta + \frac{U}{8}(1 - s_1 s_2 - s_2 s_3 - s_3 s_4 - s_1 s_4 - s_1 s_3 - s_2 s_4 + s_1 s_2 s_3 s_4)\delta^2 + \mathcal{O}(\delta^3), \quad (\text{C5})$$

$$W_{s_1 s_2; s_3 s_4} = \frac{U}{4}s_1 s_2 s_3 s_4 - \frac{U}{8}(s_1 s_2 s_3 + s_1 s_2 s_4 + s_1 s_3 s_4 + s_2 s_3 s_4)\delta + \frac{U}{16}(s_1 s_2 + s_1 s_3 + s_1 s_4 + s_2 s_3 + s_2 s_4 + s_3 s_4 - 2s_1 s_2 s_3 s_4)\delta^2 + \mathcal{O}(\delta^3). \quad (\text{C6})$$

For $\delta = 0$, the partial amplitudes reduce back to Eqs. (4.21a-4.21c). The RG flow equations for the partial amplitudes are the same as in the absence of the magnetic field and are given by Eqs. (4.24a)–(4.24c) with initial conditions (C4)–(C6). Since the differential equations for U , V , and W are identical, for the sake of argument we select the first one, copied below for the reader's convenience,

$$-\frac{d}{dL}U_{s_1 s_2; s_3 s_4}(L) = \sum_s U_{s_1 s_2; ss}(L)U_{ss; s_3 s_4}(L) \quad (\text{C7})$$

and introduce the following ansatz

$$U_{s_1 s_2; s_3 s_4}(L) = \frac{U_{s_1 s_2; s_3 s_4}(0) + a_{s_1 s_2; s_3 s_4} L}{1 + bL}, \quad (\text{C8})$$

which satisfies the initial condition. Substituting this formula into the differential equation for U and multiplying the result by $(1 + bL)^2$, we obtain an algebraic equation for $a_{s_1 s_2; s_3 s_4}$ and b

$$bU_{s_1 s_2; s_3 s_4}(0) - a_{s_1 s_2; s_3 s_4} + \sum_s (U_{s_1 s_2; ss}(0) + a_{s_1 s_2; ss} L)(U_{ss; s_3 s_4}(0) + a_{ss; s_3 s_4} L) = 0. \quad (\text{C9})$$

Grouping coefficients of a polynomial in L , we obtain the following set of equations

$$bU_{s_1 s_2; s_3 s_4}(0) - a_{s_1 s_2; s_3 s_4} + \sum_s U_{s_1 s_2; ss}(0)U_{ss; s_3 s_4}(0) = 0, \quad (\text{C10})$$

$$\sum_s [U_{s_1 s_2; ss}(0)a_{ss; s_3 s_4} + a_{s_1 s_2; ss}U_{ss; s_3 s_4}(0)] = 0, \quad (\text{C11})$$

$$\sum_s a_{s_1 s_2; ss}a_{ss; s_3 s_4} = 0, \quad (\text{C12})$$

which are not independent. Thereby, we choose two out of three equations, namely, Eq. (C10) and Eq. (C12) with the $s_1 = s_2 = s_3 = s_4 = 1$ combination of the Rashba indices, so that there are 17 equations for 17 unknown variables: $a_{s_1 s_2; s_3 s_4}$ and b . The final solutions are listed below

$$U_{\pm\pm; \pm\pm}(L) = \frac{U}{4} \frac{(1 \pm \delta)^2}{1 + (1 + \delta^2)UL/2}, \quad (\text{C13})$$

$$U_{ss; -s-s}(L) = U_{s-s; -ss}(L) = U_{s-s; s-s}(L) = \frac{U}{4} \frac{1 - \delta^2}{1 + (1 + \delta^2)UL/2}, \quad (\text{C14})$$

$$U_{\sigma(\pm\mp; \mp\mp)}(L) = \frac{U}{4} \frac{1 \mp \delta - \delta^2/2}{1 + (1 + \delta^2)UL/2}. \quad (\text{C15})$$

The same procedure is repeated to derive the V - and W -amplitudes listed below

$$V_{ss;ss}(L) = \frac{U}{2} \frac{1 - \delta^2}{1 + (1 - \delta^2)UL}, \quad (C16)$$

$$V_{ss;-s-s}(L) = -\frac{U}{2} \frac{1 - \delta^2}{1 + (1 - \delta^2)UL}, \quad (C17)$$

$$V_{s-s;-ss}(L) = -\frac{U}{2}(1 - \delta^2) - \frac{U}{2} \frac{\delta^2 UL}{1 + (1 - \delta^2)UL}, \quad (C18)$$

$$V_{s-s;s-s}(L) = \frac{U}{2}(1 + \delta^2) - \frac{U}{2} \frac{\delta^2 UL}{1 + (1 - \delta^2)UL}, \quad (C19)$$

$$V_{\sigma(\pm\mp;\mp\mp)}(L) = \pm \frac{U}{2} \frac{\delta}{1 + (1 - \delta^2)UL} \quad (C20)$$

and

$$W_{\pm\pm;\pm\pm}(L) = \frac{U}{4} \frac{(1 \mp \delta)^2}{1 + (1 + \delta^2)UL/2}, \quad (C21)$$

$$W_{ss;-s-s}(L) = W_{s-s;-ss}(L) = W_{s-s;s-s}(L) = \frac{U}{4} \frac{1 - \delta^2}{1 + (1 + \delta^2)UL/2}, \quad (C22)$$

$$W_{\sigma(\pm\mp;\mp\mp)}(L) = -\frac{U}{4} \frac{1 \pm \delta - \delta^2/2}{1 + (1 + \delta^2)UL/2}. \quad (C23)$$

Summing up all the contributions to the backscattering amplitude, obtained from the Cooper amplitude for a special choice of the momenta $\mathbf{p} = -\mathbf{k}$, i.e., for $\theta_{\mathbf{p}} - \theta_{\mathbf{k}} = \pi$, we obtain

$$\Gamma_{s_1 s_2; s_3 s_4}(\mathbf{k}, -\mathbf{k}; -\mathbf{k}, \mathbf{k}) = U_{s_1 s_2; s_3 s_4}(L) - V_{s_1 s_2; s_3 s_4}(L) + W_{s_1 s_2; s_3 s_4}(L). \quad (C24)$$

To second order in the field, the backscattering amplitudes read

$$\Gamma_{ss;ss}(\mathbf{k}, -\mathbf{k}; -\mathbf{k}, \mathbf{k}) = \left(\frac{U}{2 + UL} - \frac{U}{2(1 + UL)} \right) + \left(\frac{U}{2(1 + UL)^2} + \frac{2U}{(2 + UL)^2} \right) \delta^2, \quad (C25a)$$

$$\Gamma_{ss;-s-s}(\mathbf{k}, -\mathbf{k}; -\mathbf{k}, \mathbf{k}) = \left(\frac{U}{2 + UL} + \frac{U}{2(1 + UL)} \right) - \left(\frac{U}{2(1 + UL)^2} - \frac{2U}{(2 + UL)^2} + \frac{2U}{2 + UL} \right) \delta^2, \quad (C25b)$$

$$\Gamma_{s-s;-ss}(\mathbf{k}, -\mathbf{k}; -\mathbf{k}, \mathbf{k}) = \left(\frac{U}{2 + UL} + \frac{U}{2} \right) - \left(\frac{U}{2(1 + UL)} - \frac{2U}{(2 + UL)^2} + \frac{2U}{2 + UL} \right) \delta^2, \quad (C25c)$$

$$\Gamma_{s-s;s-s}(\mathbf{k}, -\mathbf{k}; -\mathbf{k}, \mathbf{k}) = \left(\frac{U}{2 + UL} - \frac{U}{2} \right) - \left(\frac{U}{2(1 + UL)} - \frac{2U}{(2 + UL)^2} + \frac{2U}{2 + UL} \right) \delta^2, \quad (C25d)$$

$$\Gamma_{\sigma(\pm\mp;\mp\mp)}(\mathbf{k}, -\mathbf{k}; -\mathbf{k}, \mathbf{k}) = \mp \left(\frac{U}{2(1 + UL)} + \frac{U}{2 + UL} \right) \delta. \quad (C25e)$$

For $\delta = 0$, we reproduce Eqs. (4.26-4.28). In the limit of strong renormalization in the Cooper channel, i.e., for $UL \rightarrow \infty$, the field-dependent terms in Eqs. (C25a-C25e) vanish. *A posteriori*, this explains why we could obtain the renormalized result (4.35) for χ_{zz} in the main text using only the zero-field amplitudes.

b. Renormalization of χ_{zz} in the Cooper channel

As in Sec. IV C 3, the thermodynamic potential in the presence of Cooper renormalization is obtained by substituting the renormalized scattering amplitudes (C25a-C25e) into Eq. (4.32)

$$\delta\Xi_{zz} = -\frac{1}{2}T \sum_{\Omega} \int \frac{qdq}{2\pi} \left[\frac{1}{2} \left(\frac{U}{2+UL} - \frac{U}{2} \right)^2 (\Pi_{+-}^2 + \Pi_{-+}^2 - 2\Pi_0^2) + \left(\frac{U}{2+UL} + \frac{U}{2(1+UL)} \right)^2 (\Pi_{+-}\Pi_{-+} - \Pi_0^2) \right. \quad (C26a)$$

$$\left. - \frac{U^2(16 + 32UL + 22U^2L^2 + 6U^2L^3 + U^4L^4)}{4(1+UL)^2(2+UL)^2} \Pi_0^2 \right] \\ - \frac{\Delta^2}{\alpha^2 k_F^2} T \sum_{\Omega} \int \frac{qdq}{2\pi} \left[\left(\frac{U}{2(1+UL)} + \frac{U}{2+UL} \right)^2 \Pi_0(\Pi_{+-} + \Pi_{-+} - 2\Pi_0) \right. \quad (C26b)$$

$$+ \frac{1}{2} \left(\frac{U}{2(1+UL)} - \frac{2U}{(2+UL)^2} + \frac{2U}{2+UL} \right) \left(\frac{U}{2+UL} - \frac{U}{2} \right) (\Pi_{+-}^2 + \Pi_{-+}^2 - 2\Pi_0^2) \\ - \left(\frac{U}{2(1+UL)^2} - \frac{2U}{(2+UL)^2} + \frac{2U}{2+UL} \right) \left(\frac{U}{2+UL} + \frac{U}{2(1+UL)} \right) (\Pi_{+-}\Pi_{-+} - \Pi_0^2) \\ \left. - \frac{U^4L^2(12 + 18UL + 7U^2L^2)}{2(1+UL)^2(2+UL)^3} \Pi_0^2 \right]. \quad (C26c)$$

The first part of $\delta\Xi_{zz}$ (C26a) came from the field-independent terms in the scattering amplitudes. This part depends on the magnetic field through the combinations of the polarization bubbles. The second part (C26c) already contains a field-dependent prefactor (Δ^2) resulting from the field-dependent terms in the scattering amplitudes. Therefore, the polarization bubbles in this part can be evaluated in zero field. The integrals over q along with the summation over the Matsubara frequency Ω have already been performed in Sec. III. Note that in each square bracket the last term (proportional to Π_0^2) depends neither on the field nor on the SOI. In fact, it can be shown^{9,29} that this formally divergent contribution has a cubic dependence on temperature, $T \sum_{\Omega} \int qdq \Pi_0^2 \propto T^3$, thus it adds a higher order correction and can be dropped. The final result reads

$$\delta\Xi_{zz} = -\frac{T^3}{8\pi v_F^2} \left(\frac{mU}{2\pi} \right)^2 \left[\left(\frac{1}{2+UL} - \frac{1}{2} \right)^2 \mathcal{F} \left(\frac{2\sqrt{\alpha^2 k_F^2 + \Delta^2}}{T} \right) \right] \\ - \frac{\Delta^2}{\alpha^2 k_F^2} \frac{T^3}{2\pi v_F^2} \left(\frac{mU}{2\pi} \right)^2 \left[\left(\frac{1}{2(1+UL)} + \frac{1}{2+UL} \right)^2 \mathcal{F} \left(\frac{|\alpha|k_F}{T} \right) \right. \quad (C27) \\ \left. + \frac{1}{2} \left(\frac{1}{2(1+UL)} - \frac{2}{(2+UL)^2} + \frac{2}{2+UL} \right) \left(\frac{1}{2+UL} - \frac{1}{2} \right) \mathcal{F} \left(\frac{2|\alpha|k_F}{T} \right) \right].$$

The spin susceptibility is obtained by expanding Eq. (C27) further to order Δ^2 . We also need to recall that our treatment of Cooper renormalization is only valid for $T \ll |\alpha|k_F$, because we kept only the T - but not α -dependent Cooper logarithms (see the discussion at the end of Sec. IV C 1). Therefore, the function \mathcal{F} and its derivative should be replaced by their large-argument forms, Eq. (3.27). Doing so, we obtain the final result for the nonanalytic part of the χ_{zz} , presented in Eq. (4.37) in the main text.

2. In-plane magnetic field

a. RG flow of the scattering amplitudes

For the in-plane magnetic field, $\mathbf{B} = B\mathbf{e}_x$, the RG flow of the scattering amplitudes is more cumbersome because the eigenvectors of Hamiltonian (1.1) depend in a complicated way on the angle between the magnetic field and the electron momentum

$$|\mathbf{k}, s\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} s\bar{\Delta}_{\mathbf{k}}/(\Delta - ie^{i\theta_{\mathbf{k}}}\alpha k) \\ 1 \end{pmatrix}, \quad (C28)$$

where $\bar{\Delta}_{\mathbf{k}} \equiv (\Delta^2 + 2\alpha k \Delta \sin \theta_{\mathbf{k}} + \alpha^2 k^2)^{-1/2}$ is the effective Zeeman energy. For that reason, the (double) Fourier series of the scattering amplitude (4.17) in the angles $\theta_{\mathbf{k}}$ and $\theta_{\mathbf{p}}$ contains infinitely many harmonics:

$$\Gamma_{s_1 s_2; s_3 s_4}^{(1)}(\mathbf{k}, -\mathbf{k}; \mathbf{p}, -\mathbf{p}) = \sum_{m, n=-\infty}^{\infty} \Gamma_{s_1 s_2; s_3 s_4}^{(1)\{m, n\}} e^{im\theta_{\mathbf{p}}} e^{in\theta_{\mathbf{k}}}, \quad (\text{C29})$$

To second order in the field, however, the number of nonvanishing harmonics is limited to 15. Indeed, expanding the eigenvector (C28) to second order in $\delta \equiv \Delta/|\alpha|k_F$ as

$$|\mathbf{k}, s\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} s i e^{-i\theta_{\mathbf{k}}} [1 - i \cos \theta_{\mathbf{k}} \delta - (\cos^2 \theta_{\mathbf{k}} - i \sin 2\theta_{\mathbf{k}}) \delta^2 / 2] \\ 1 \end{pmatrix} + \mathcal{O}(\delta^3) \quad (\text{C30})$$

and substituting Eq. (C30) into the scattering amplitude (4.17) with $\mathbf{k}' = -\mathbf{k}$ and $\mathbf{p}' = -\mathbf{p}$, we obtain

$$\begin{aligned} \Gamma_{s_1 s_2; s_3 s_4}^{(1)}(\mathbf{k}, -\mathbf{k}; \mathbf{p}, -\mathbf{p}) &= U_{s_1 s_2; s_3 s_4} + V_{s_1 s_2; s_3 s_4} e^{i\theta_{\mathbf{p}}} e^{-i\theta_{\mathbf{k}}} + W_{s_1 s_2; s_3 s_4} e^{2i\theta_{\mathbf{p}}} e^{-2i\theta_{\mathbf{k}}} \\ &+ A_{s_1 s_2; s_3 s_4} e^{i\theta_{\mathbf{p}}} e^{i\theta_{\mathbf{k}}} + B_{s_1 s_2; s_3 s_4} e^{-i\theta_{\mathbf{p}}} e^{-i\theta_{\mathbf{k}}} + F_{s_1 s_2; s_3 s_4} e^{2i\theta_{\mathbf{p}}} e^{-i\theta_{\mathbf{k}}} + G_{s_1 s_2; s_3 s_4} e^{i\theta_{\mathbf{p}}} e^{-2i\theta_{\mathbf{k}}} \\ &+ H_{s_1 s_2; s_3 s_4} e^{i\theta_{\mathbf{p}}} + J_{s_1 s_2; s_3 s_4} e^{-i\theta_{\mathbf{k}}} + L_{s_1 s_2; s_3 s_4} e^{2i\theta_{\mathbf{p}}} + M_{s_1 s_2; s_3 s_4} e^{-2i\theta_{\mathbf{k}}} \\ &+ P_{s_1 s_2; s_3 s_4} e^{3i\theta_{\mathbf{p}}} e^{-i\theta_{\mathbf{k}}} + Q_{s_1 s_2; s_3 s_4} e^{i\theta_{\mathbf{p}}} e^{-3i\theta_{\mathbf{k}}} + R_{s_1 s_2; s_3 s_4} e^{4i\theta_{\mathbf{p}}} e^{-2i\theta_{\mathbf{k}}} + S_{s_1 s_2; s_3 s_4} e^{2i\theta_{\mathbf{p}}} e^{-4i\theta_{\mathbf{k}}} + \mathcal{O}(\delta^3). \end{aligned} \quad (\text{C31})$$

Here,

$$U_{s_1 s_2; s_3 s_4} = \Gamma_{s_1 s_2; s_3 s_4}^{(1)\{0,0\}} = (U/16)[4 + (s_1 s_3 + s_2 s_4) \delta^2], \quad (\text{C32})$$

$$V_{s_1 s_2; s_3 s_4} = \Gamma_{s_1 s_2; s_3 s_4}^{(1)\{1,-1\}} = (U/8)(s_1 s_3 + s_2 s_4)(2 - \delta^2), \quad (\text{C33})$$

$$W_{s_1 s_2; s_3 s_4} = \Gamma_{s_1 s_2; s_3 s_4}^{(1)\{2,-2\}} = (U/16)[4s_1 s_2 s_3 s_4 + (s_1 s_3 + s_2 s_4) \delta^2], \quad (\text{C34})$$

$$A_{s_1 s_2; s_3 s_4} = \Gamma_{s_1 s_2; s_3 s_4}^{(1)\{1,1\}} = (U/32)(s_1 s_3 + s_2 s_4) \delta^2 \quad (\text{C35})$$

$$B_{s_1 s_2; s_3 s_4} = \Gamma_{s_1 s_2; s_3 s_4}^{(1)\{-1,-1\}} = A_{s_1 s_2; s_3 s_4}, \quad (\text{C36})$$

$$F_{s_1 s_2; s_3 s_4} = \Gamma_{s_1 s_2; s_3 s_4}^{(1)\{2,-1\}} = (U/8)i(s_1 s_3 - s_2 s_4) \delta, \quad (\text{C37})$$

$$G_{s_1 s_2; s_3 s_4} = \Gamma_{s_1 s_2; s_3 s_4}^{(1)\{1,-2\}} = -F_{s_1 s_2; s_3 s_4}, \quad (\text{C38})$$

$$H_{s_1 s_2; s_3 s_4} = \Gamma_{s_1 s_2; s_3 s_4}^{(1)\{1,0\}} = -F_{s_1 s_2; s_3 s_4}, \quad (\text{C39})$$

$$J_{s_1 s_2; s_3 s_4} = \Gamma_{s_1 s_2; s_3 s_4}^{(1)\{0,-1\}} = F_{s_1 s_2; s_3 s_4}, \quad (\text{C40})$$

$$L_{s_1 s_2; s_3 s_4} = \Gamma_{s_1 s_2; s_3 s_4}^{(1)\{2,0\}} = (U/16)(s_1 s_3 + s_2 s_4 + 2s_1 s_2 s_3 s_4) \delta^2, \quad (\text{C41})$$

$$M_{s_1 s_2; s_3 s_4} = \Gamma_{s_1 s_2; s_3 s_4}^{(1)\{0,-2\}} = L_{s_1 s_2; s_3 s_4}, \quad (\text{C42})$$

$$P_{s_1 s_2; s_3 s_4} = \Gamma_{s_1 s_2; s_3 s_4}^{(1)\{3,-1\}} = -3A_{s_1 s_2; s_3 s_4}, \quad (\text{C43})$$

$$Q_{s_1 s_2; s_3 s_4} = \Gamma_{s_1 s_2; s_3 s_4}^{(1)\{1,-3\}} = P_{s_1 s_2; s_3 s_4}, \quad (\text{C44})$$

$$R_{s_1 s_2; s_3 s_4} = \Gamma_{s_1 s_2; s_3 s_4}^{(1)\{4,-2\}} = -(U/8)s_1 s_2 s_3 s_4 \delta^2, \quad (\text{C45})$$

$$S_{s_1 s_2; s_3 s_4} = \Gamma_{s_1 s_2; s_3 s_4}^{(1)\{2,-4\}} = R_{s_1 s_2; s_3 s_4} \quad (\text{C46})$$

The second-order amplitude is derived from Eq. (4.22) with $n = 2$

$$\begin{aligned} \Gamma_{s_1 s_2; s_3 s_4}^{(2)}(\mathbf{k}, -\mathbf{k}; \mathbf{p}, -\mathbf{p}) &= -L \sum_s \{ U_{s_1 s_2; ss} U_{ss; s_3 s_4} + H_{s_1 s_2; ss} J_{ss; s_3 s_4} \\ &+ [V_{s_1 s_2; ss} V_{ss; s_3 s_4} + F_{s_1 s_2; ss} G_{ss; s_3 s_4} + J_{s_1 s_2; ss} H_{ss; s_3 s_4}] e^{i\theta_{\mathbf{p}}} e^{-i\theta_{\mathbf{k}}} + [W_{s_1 s_2; ss} W_{ss; s_3 s_4} + G_{s_1 s_2; ss} F_{ss; s_3 s_4}] e^{2i\theta_{\mathbf{p}}} e^{-2i\theta_{\mathbf{k}}} \\ &+ A_{s_1 s_2; ss} V_{ss; s_3 s_4} e^{i\theta_{\mathbf{p}}} e^{i\theta_{\mathbf{k}}} + V_{s_1 s_2; ss} B_{ss; s_3 s_4} e^{-i\theta_{\mathbf{p}}} e^{-i\theta_{\mathbf{k}}} + [V_{s_1 s_2; ss} F_{ss; s_3 s_4} + F_{s_1 s_2; ss} W_{ss; s_3 s_4}] e^{2i\theta_{\mathbf{p}}} e^{-i\theta_{\mathbf{k}}} \\ &+ [W_{s_1 s_2; ss} G_{ss; s_3 s_4} + G_{s_1 s_2; ss} V_{ss; s_3 s_4}] e^{i\theta_{\mathbf{p}}} e^{-2i\theta_{\mathbf{k}}} + [H_{s_1 s_2; ss} V_{ss; s_3 s_4} + U_{s_1 s_2; ss} H_{ss; s_3 s_4}] e^{i\theta_{\mathbf{p}}} \\ &+ [V_{s_1 s_2; ss} J_{ss; s_3 s_4} + J_{s_1 s_2; ss} U_{ss; s_3 s_4}] e^{-i\theta_{\mathbf{k}}} + [H_{s_1 s_2; ss} F_{ss; s_3 s_4} + U_{s_1 s_2; ss} L_{ss; s_3 s_4} + L_{s_1 s_2; ss} W_{ss; s_3 s_4}] e^{2i\theta_{\mathbf{p}}} \\ &+ [W_{s_1 s_2; ss} M_{ss; s_3 s_4} + G_{s_1 s_2; ss} J_{ss; s_3 s_4} + M_{s_1 s_2; ss} U_{ss; s_3 s_4}] e^{-2i\theta_{\mathbf{k}}} + V_{s_1 s_2; ss} P_{ss; s_3 s_4} e^{3i\theta_{\mathbf{p}}} e^{-i\theta_{\mathbf{k}}} \\ &+ Q_{s_1 s_2; ss} V_{ss; s_3 s_4} e^{i\theta_{\mathbf{p}}} e^{-3i\theta_{\mathbf{k}}} + W_{s_1 s_2; ss} R_{ss; s_3 s_4} e^{4i\theta_{\mathbf{p}}} e^{-2i\theta_{\mathbf{k}}} + S_{s_1 s_2; ss} W_{ss; s_3 s_4} e^{2i\theta_{\mathbf{p}}} e^{-4i\theta_{\mathbf{k}}} \} + \mathcal{O}(\delta^3), \end{aligned} \quad (\text{C47})$$

where $L = (m/2\pi) \ln(\Lambda/T)$. The second-order amplitude contains the same combinations of the harmonics $e^{im\theta_{\mathbf{p}}}$ and $e^{in\theta_{\mathbf{k}}}$ as the first-order amplitude, which proves the group property. The RG flow equations are obtained by replacing the left-hand side of Eq. (C47) by the bare amplitude, letting the coefficients $U \dots S$ to depend on L , and differentiating with respect to L :

$$-\frac{d}{dL}U_{s_1s_2;s_3s_4}(L) = U_{s_1s_2;ss}(L)U_{ss;s_3s_4}(L) + H_{s_1s_2;ss}(L)J_{ss;s_3s_4}(L), \quad (\text{C48})$$

$$-\frac{d}{dL}V_{s_1s_2;s_3s_4}(L) = V_{s_1s_2;ss}(L)V_{ss;s_3s_4}(L) + F_{s_1s_2;ss}(L)G_{ss;s_3s_4}(L) + J_{s_1s_2;ss}(L)H_{ss;s_3s_4}(L), \quad (\text{C49})$$

$$-\frac{d}{dL}W_{s_1s_2;s_3s_4}(L) = W_{s_1s_2;ss}(L)W_{ss;s_3s_4}(L) + G_{s_1s_2;ss}(L)F_{ss;s_3s_4}(L), \quad (\text{C50})$$

$$-\frac{d}{dL}A_{s_1s_2;s_3s_4}(L) = A_{s_1s_2;ss}(L)V_{ss;s_3s_4}(L), \quad (\text{C51})$$

$$-\frac{d}{dL}B_{s_1s_2;s_3s_4}(L) = V_{s_1s_2;ss}(L)B_{ss;s_3s_4}(L), \quad (\text{C52})$$

$$-\frac{d}{dL}F_{s_1s_2;s_3s_4}(L) = V_{s_1s_2;ss}(L)F_{ss;s_3s_4}(L) + F_{s_1s_2;ss}(L)W_{ss;s_3s_4}(L), \quad (\text{C53})$$

$$-\frac{d}{dL}G_{s_1s_2;s_3s_4}(L) = W_{s_1s_2;ss}(L)G_{ss;s_3s_4}(L) + G_{s_1s_2;ss}(L)V_{ss;s_3s_4}(L), \quad (\text{C54})$$

$$-\frac{d}{dL}H_{s_1s_2;s_3s_4}(L) = H_{s_1s_2;ss}(L)V_{ss;s_3s_4}(L) + U_{s_1s_2;ss}(L)H_{ss;s_3s_4}(L), \quad (\text{C55})$$

$$-\frac{d}{dL}J_{s_1s_2;s_3s_4}(L) = V_{s_1s_2;ss}(L)J_{ss;s_3s_4}(L) + J_{s_1s_2;ss}(L)U_{ss;s_3s_4}(L), \quad (\text{C56})$$

$$-\frac{d}{dL}L_{s_1s_2;s_3s_4}(L) = H_{s_1s_2;ss}(L)F_{ss;s_3s_4}(L) + U_{s_1s_2;ss}(L)L_{ss;s_3s_4}(L) + L_{s_1s_2;ss}(L)W_{ss;s_3s_4}(L), \quad (\text{C57})$$

$$-\frac{d}{dL}M_{s_1s_2;s_3s_4}(L) = W_{s_1s_2;ss}(L)M_{ss;s_3s_4}(L) + G_{s_1s_2;ss}(L)J_{ss;s_3s_4}(L) + M_{s_1s_2;ss}(L)U_{ss;s_3s_4}(L), \quad (\text{C58})$$

$$-\frac{d}{dL}P_{s_1s_2;s_3s_4}(L) = V_{s_1s_2;ss}(L)P_{ss;s_3s_4}(L), \quad (\text{C59})$$

$$-\frac{d}{dL}Q_{s_1s_2;s_3s_4}(L) = Q_{s_1s_2;ss}(L)V_{ss;s_3s_4}(L), \quad (\text{C60})$$

$$-\frac{d}{dL}R_{s_1s_2;s_3s_4}(L) = W_{s_1s_2;ss}(L)R_{ss;s_3s_4}(L), \quad (\text{C61})$$

$$-\frac{d}{dL}S_{s_1s_2;s_3s_4}(L) = S_{s_1s_2;ss}(L)W_{ss;s_3s_4}(L), \quad (\text{C62})$$

where summation over the repeated index s is implied. The initial conditions are given by $X_{s_1,s_2;s_3,s_4}(0) = X_{s_1,s_2;s_3,s_4}$ with $X = U \dots S$. Since it is very difficult to solve this system of differential equations analytically, a new approach is required. In what follows, we will determine $U(L), \dots, S(L)$ for a few lowest orders in the “RG time” L and then make a guess for a form of an arbitrary-order term. The RG equations will then provide a necessary check as to whether our guess, based on the perturbative calculation, gives a correct answer.

A few lowest order amplitudes can be derived perturbatively from Eq. (4.22), copied here for the reader’s convenience

$$\Gamma_{s_1s_2;s_4s_3}^{(j)}(\mathbf{k}, -\mathbf{k}; \mathbf{p}, -\mathbf{p})(L) = -L \sum_s \int_0^{2\pi} \frac{d\theta_l}{2\pi} \Gamma_{s_1s_2;ss}^{(j-1)}(\mathbf{k}, -\mathbf{k}; \mathbf{l}, -\mathbf{l}) \Gamma_{ss;s_4s_3}^{(1)}(\mathbf{l}, -\mathbf{l}; \mathbf{p}, -\mathbf{p}) \quad (\text{C63})$$

with $j \geq 2$ standing for order of the perturbation theory. Since the scattering amplitudes depend on angles $\theta_{\mathbf{k}}$ and $\theta_{\mathbf{p}}$, they can be decomposed order by order into the Fourier series

$$\Gamma_{s_1s_2;s_3s_4}^{(j)}(\mathbf{k}, -\mathbf{k}; \mathbf{p}, -\mathbf{p}) = \sum_{m,n=-\infty}^{\infty} \Gamma_{s_1s_2;s_3s_4}^{(j)\{m,n\}} e^{mi\theta_{\mathbf{p}}} e^{ni\theta_{\mathbf{k}}}, \quad (\text{C64})$$

where the coefficients in front of $e^{im\theta_{\mathbf{p}}} e^{in\theta_{\mathbf{k}}}$ are determined using the orthogonality property

$$\Gamma_{s_1s_2;s_4s_3}^{(j)\{m,n\}}(\mathbf{k}, -\mathbf{k}; \mathbf{p}, -\mathbf{p}) = \int_0^{2\pi} \frac{d\theta_{\mathbf{k}}}{2\pi} \int_0^{2\pi} \frac{d\theta_{\mathbf{p}}}{2\pi} \Gamma_{s_1s_2;s_4s_3}^{(j)}(\mathbf{k}, -\mathbf{k}; \mathbf{p}, -\mathbf{p}) e^{-im\theta_{\mathbf{p}}} e^{-in\theta_{\mathbf{k}}}. \quad (\text{C65})$$

Resumming the coefficients of $e^{im\theta_{\mathbf{p}}}e^{in\theta_{\mathbf{k}}}$ to infinite order (with m and n being fixed)

$$\Gamma_{s_1 s_2; s_4 s_3}^{(\infty)\{m,n\}}(\mathbf{k}, -\mathbf{k}; \mathbf{p}, -\mathbf{p}) = \sum_{j=1}^{\infty} \Gamma_{s_1 s_2; s_4 s_3}^{(j)\{m,n\}}(\mathbf{k}, -\mathbf{k}; \mathbf{p}, -\mathbf{p}) \quad (\text{C66})$$

we can find the renormalized amplitudes. For each combination of partial harmonics, which occurs to second order in the magnetic field, we derive explicitly the scattering amplitudes up to seventh order in the Cooper channel renormalization parameter UL and then make a guess for general j -th order amplitude. The final result is obtained by resumming these amplitudes to infinite order and then substituted into the RG flow equations to check the correctness of our guess. In all cases, the guess turns out to be correct. All *nonzero* RG charges as well as their large L limits are listed below. We begin with the $n = m = 0$ and $n = m = 1$ harmonics, given by

$$U_{ss;ss}(L) = \frac{U}{2} \frac{1}{2+UL} + \frac{U}{8} \delta^2 = \frac{U}{8} \delta^2 + \mathcal{O}(\ln^{-1} T), \quad (\text{C67})$$

$$U_{ss;-s-s}(L) = \frac{U}{2} \frac{1}{2+UL} - \frac{U}{8} \delta^2 = -\frac{U}{8} \delta^2 + \mathcal{O}(\ln^{-1} T), \quad (\text{C68})$$

$$U_{s-s;-ss}(L) = \frac{U}{2} \frac{1}{2+UL} - \frac{U}{8} \frac{1}{1+UL} \delta^2 = \mathcal{O}(\ln^{-1} T), \quad (\text{C69})$$

$$U_{s-s;s-s}(L) = \frac{U}{2} \frac{1}{2+UL} + \frac{U}{8} \frac{1}{1+UL} \delta^2 = \mathcal{O}(\ln^{-1} T), \quad (\text{C70})$$

$$U_{\sigma(\pm\mp;\mp\mp)}(L) = \frac{U}{2} \frac{1}{2+UL} = \mathcal{O}(\ln^{-1} T), \quad (\text{C71})$$

$$V_{ss;ss}(L) = \frac{U}{2} \frac{1}{1+UL} - \frac{U}{4} \frac{1}{(1+UL)^2} \delta^2 = \mathcal{O}(\ln^{-1} T), \quad (\text{C72})$$

$$V_{ss;-s-s}(L) = -\frac{U}{2} \frac{1}{1+UL} + \frac{U}{4} \frac{1}{(1+UL)^2} \delta^2 = \mathcal{O}(\ln^{-1} T), \quad (\text{C73})$$

$$V_{s-s;-ss}(L) = -\frac{U}{2} + \frac{U}{4} (1+UL) \delta^2 = -\frac{U}{2} + \mathcal{O}(U^2), \quad (\text{C74})$$

$$V_{s-s;s-s}(L) = \frac{U}{2} - \frac{U}{4} (1+UL) \delta^2 = \frac{U}{2} + \mathcal{O}(U^2). \quad (\text{C75})$$

An important remark should be made at this point: in addition to amplitudes which flow either to zero or to finite values at low temperatures, there are also amplitudes which grow logarithmically at low temperatures, namely, the amplitudes in Eqs.(C74) and (C75). This peculiar feature, which occurs only in the presence of both the SOI and in-plane magnetic field, may indicate a phase transition below certain field-dependent temperature or it may be an artifact of the expansion to lowest order in δ^2 . In the derivation of the spin susceptibility that follows in App. C 2 b, we assume that the electron gas is far above the temperature below which the instability becomes important, i.e., that $UL \ll 1/\delta^2$, so that the effect of the instability can be neglected but the nonperturbative regime of Cooper renormalization, where $1 \ll UL \ll 1/\delta^2$, can still be accessed.

The remaining harmonics are

$$W_{ss;ss}(L) = U_{ss;ss}(L), \quad (\text{C76})$$

$$W_{ss;-s-s}(L) = U_{ss;-s-s}(L), \quad (\text{C77})$$

$$W_{s-s;-ss}(L) = U_{s-s;-ss}(L), \quad (\text{C78})$$

$$W_{s-s;s-s}(L) = U_{s-s;s-s}(L), \quad (\text{C79})$$

$$W_{\sigma(\pm\mp;\mp\mp)}(L) = -U_{\sigma(\pm\mp;\mp\mp)}(L), \quad (\text{C80})$$

$$A_{ss;ss}(L) = -A_{ss;-s-s}(L) = \frac{U}{16} \frac{1}{1+UL} \delta^2 = \mathcal{O}(\ln^{-1} T), \quad (\text{C81})$$

$$A_{s-s;-ss}(L) = -A_{s-s;s-s}(L) = -\frac{U}{16} \delta^2 \quad (\text{C82})$$

$$B_{s_1 s_2; s_3 s_4}(L) = A_{s_1 s_2; s_3 s_4}(L). \quad (\text{C83})$$

$$F_{\pm\mp;\mp\mp}(L) = -\frac{U}{4}i\delta, \quad (C84)$$

$$F_{\mp\pm;\mp\mp}(L) = \frac{U}{4}i\delta, \quad (C85)$$

$$F_{\mp\mp;\pm\mp}(L) = -\frac{U}{4} \frac{1}{1+UL} i\delta = \mathcal{O}(\ln^{-1} T), \quad (C86)$$

$$F_{\mp\mp;\mp\pm}(L) = \frac{U}{4} \frac{1}{1+UL} i\delta = \mathcal{O}(\ln^{-1} T), \quad (C87)$$

$$G_{\pm\mp;\mp\mp}(L) = \frac{U}{4} \frac{1}{1+UL} i\delta = \mathcal{O}(\ln^{-1} T), \quad (C88)$$

$$G_{\mp\pm;\mp\mp}(L) = -\frac{U}{4} \frac{1}{1+UL} i\delta = \mathcal{O}(\ln^{-1} T), \quad (C89)$$

$$G_{\mp\mp;\pm\mp}(L) = \frac{U}{4} i\delta, \quad (C90)$$

$$G_{\mp\mp;\mp\pm}(L) = -\frac{U}{4} i\delta, \quad (C91)$$

$$H_{s_1 s_2; s_3 s_4}(L) = G_{s_1 s_2; s_3 s_4}(L) \quad (C92)$$

$$J_{s_1 s_2; s_3 s_4}(L) = F_{s_1 s_2; s_3 s_4}(L) \quad (C93)$$

$$L_{ss;ss}(L) = \frac{U}{2} \frac{1}{(2+UL)^2} + \frac{U}{8} \delta^2 = \frac{U}{8} \delta^2 + \mathcal{O}(\ln^{-1} T), \quad (C94)$$

$$L_{ss;-s-s}(L) = \frac{U}{2} \frac{1}{(2+UL)^2} - \frac{U}{8} \delta^2 = -\frac{U}{8} \delta^2 + \mathcal{O}(\ln^{-1} T), \quad (C95)$$

$$L_{s-s;-ss}(L) = \frac{U}{8} \frac{(4+3UL)UL}{(1+UL)(2+UL)^2} \delta^2 = \mathcal{O}(\ln^{-1} T), \quad (C96)$$

$$L_{s-s;s-s}(L) = \frac{U}{8} \frac{8+12UL+5U^2L^2}{(1+UL)(2+UL)^2} \delta^2 = \mathcal{O}(\ln^{-1} T), \quad (C97)$$

$$L_{\pm\mp;\mp\mp}(L) = L_{\mp\pm;\mp\mp}(L) = -\frac{U}{2} \frac{1+UL}{(2+UL)^2} \delta^2 = \mathcal{O}(\ln^{-1} T), \quad (C98)$$

$$L_{\mp\mp;\pm\mp}(L) = L_{\mp\mp;\mp\pm}(L) = -\frac{U}{2} \frac{1}{(2+UL)^2} \delta^2 = \mathcal{O}(\ln^{-1} T). \quad (C99)$$

$$M_{ss;ss}(L) = L_{ss;ss}(L), \quad (C100)$$

$$M_{ss;-s-s}(L) = L_{ss;-s-s}(L), \quad (C101)$$

$$M_{s-s;-ss}(L) = L_{s-s;-ss}(L), \quad (C102)$$

$$M_{s-s;s-s}(L) = L_{s-s;s-s}(L), \quad (C103)$$

$$M_{\pm\mp;\mp\mp}(L) = L_{\mp\pm;\mp\mp}(L) = -\frac{U}{2} \frac{1}{(2+UL)^2} \delta^2 = \mathcal{O}(\ln^{-1} T), \quad (C104)$$

$$M_{\mp\mp;\pm\mp}(L) = L_{\mp\mp;\mp\pm}(L) = -\frac{U}{2} \frac{1+UL}{(2+UL)^2} \delta^2 = \mathcal{O}(\ln^{-1} T), \quad (C105)$$

$$P_{s_1 s_2; s_3 s_4}(L) = Q_{s_1 s_2; s_3 s_4}(L) = -3A_{s_1 s_2; s_3 s_4}(L), \quad (C106)$$

$$R_{ss;ss}(L) = R_{ss;-s-s}(L) = R_{s-s;-ss}(L) = R_{s-s;s-s}(L) = -\frac{U}{4} \frac{1}{2+UL} \delta^2 = \mathcal{O}(\ln^{-1} T), \quad (C107)$$

$$R_{\sigma(\pm\mp;\mp\mp)}(L) = \frac{U}{4} \frac{1}{2+UL} \delta^2 = \mathcal{O}(\ln^{-1} T), \quad (C108)$$

$$S_{s_1 s_2; s_3 s_4}(L) = R_{s_1 s_2; s_3 s_4}(L). \quad (C109)$$

It can be readily verified that all the amplitudes satisfy RG equations (C48)–(C62) with initial conditions (C32)–(C46 up to $\mathcal{O}(\delta^3)$ accuracy.

Finally, the renormalized scattering amplitude is given by

$$\begin{aligned} \Gamma_{s_1 s_2; s_3 s_4}(\mathbf{k}, -\mathbf{k}; \mathbf{p}, -\mathbf{p}) = & U_{s_1 s_2; s_3 s_4}(L) + V_{s_1 s_2; s_3 s_4}(L)e^{i\theta_{\mathbf{p}}}e^{-i\theta_{\mathbf{k}}} + W_{s_1 s_2; s_3 s_4}(L)e^{2i\theta_{\mathbf{p}}}e^{-2i\theta_{\mathbf{k}}} \\ & + A_{s_1 s_2; s_3 s_4}(L)e^{i\theta_{\mathbf{p}}}e^{i\theta_{\mathbf{k}}} + B_{s_1 s_2; s_3 s_4}(L)e^{-i\theta_{\mathbf{p}}}e^{-i\theta_{\mathbf{k}}} + F_{s_1 s_2; s_3 s_4}(L)e^{2i\theta_{\mathbf{p}}}e^{-i\theta_{\mathbf{k}}} + G_{s_1 s_2; s_3 s_4}(L)e^{i\theta_{\mathbf{p}}}e^{-2i\theta_{\mathbf{k}}} \\ & + H_{s_1 s_2; s_3 s_4}(L)e^{i\theta_{\mathbf{p}}} + J_{s_1 s_2; s_3 s_4}(L)e^{-i\theta_{\mathbf{k}}} + L_{s_1 s_2; s_3 s_4}(L)e^{2i\theta_{\mathbf{p}}} + M_{s_1 s_2; s_3 s_4}(L)e^{-2i\theta_{\mathbf{k}}} \\ & + P_{s_1 s_2; s_3 s_4}(L)e^{3i\theta_{\mathbf{p}}}e^{-i\theta_{\mathbf{k}}} + Q_{s_1 s_2; s_3 s_4}(L)e^{i\theta_{\mathbf{p}}}e^{-3i\theta_{\mathbf{k}}} + R_{s_1 s_2; s_3 s_4}(L)e^{4i\theta_{\mathbf{p}}}e^{-2i\theta_{\mathbf{k}}} + S_{s_1 s_2; s_3 s_4}(L)e^{2i\theta_{\mathbf{p}}}e^{-4i\theta_{\mathbf{k}}}. \end{aligned} \quad (C110)$$

b. Renormalization of χ_{xx}

As for the transverse-field case, the free energy for the in-plane magnetic field is found by replacing the bare interaction U in Eq. (3.38) by the renormalized vertex Γ

$$\delta\Xi_{xx} = -\frac{1}{4} \int_0^{2\pi} \frac{d\theta_{\mathbf{k}}}{2\pi} T \sum_{\Omega} \sum_{\{s_i\}} \int_0^{\infty} \frac{qdq}{2\pi} \Gamma_{s_1 s_4; s_3 s_2}(\mathbf{k}, -\mathbf{k}; -\mathbf{k}, \mathbf{k}) \Gamma_{s_3 s_2; s_1 s_4}(-\mathbf{k}, \mathbf{k}; \mathbf{k}, -\mathbf{k}) \Pi_{s_1 s_2}^{+\mathbf{k}_F} \Pi_{s_3 s_4}^{-\mathbf{k}_F}, \quad (C111)$$

where $\Pi_{ss'}^{\pm\mathbf{k}_F}$ given by Eq. (3.39) depends on the direction of the electron momentum with respect to the magnetic field.

A general formula for $\delta\Xi_{xx}$ is very complicated; however, in the regime of strong Cooper renormalization, i.e. for $1 \ll UL \ll 1/\delta^2$, there are only a few partial amplitudes which survive the downward renormalization. Keeping only these partial amplitudes in Γ , we obtain for the thermodynamic potential

$$\begin{aligned} \delta\Xi_{xx} = & -\frac{U^2}{16} \int_0^{2\pi} \frac{d\theta_{\mathbf{k}}}{2\pi} T \sum_{\Omega} \int \frac{qdq}{2\pi} [(\Pi_{-+}^{+\mathbf{k}_F} \Pi_{-+}^{-\mathbf{k}_F} + \Pi_{+-}^{+\mathbf{k}_F} \Pi_{+-}^{-\mathbf{k}_F} - 2\Pi_0^2) + 4\Pi_0^2] \\ & - \frac{\Delta^2}{\alpha^2 k_F^2} \frac{U^2}{8} T \sum_{\Omega} \int \frac{qdq}{2\pi} [\Pi_0(\Pi_{-+} + \Pi_{+-} - 2\Pi_0) + \Pi_0^2], \end{aligned} \quad (C112)$$

where the angular dependence of the polarization bubbles in the second line was neglected because of an overall factor Δ^2 originating from the scattering amplitudes, and the angular integral in those terms was readily performed. On the other hand, the field dependence in the first term is exclusively due to the bubbles, hence the angular integration has to be carried out last. The integrals over the momentum and frequency yield

$$\delta\Xi_{xx} = -\frac{T^3}{8\pi v_F^2} \left(\frac{mU}{4\pi}\right)^2 \left[\int_0^{2\pi} \frac{d\theta_{\mathbf{k}}}{2\pi} \mathcal{F}\left(\frac{\bar{\Delta}_{\mathbf{k}_F} + \bar{\Delta}_{-\mathbf{k}_F}}{T}\right) + 2\frac{\Delta^2}{\alpha^2 k_F^2} \mathcal{F}\left(\frac{|\alpha|k_F}{T}\right) \right]. \quad (C113)$$

Expanding $\mathcal{F}(x) \approx x^3/3$ for $x \gg 1$ and differentiating with respect to the field twice, we obtain for the nonanalytic part of the spin susceptibility

$$\delta\chi_{xx} = \frac{1}{3} \chi_0 \left(\frac{mU}{4\pi}\right)^2 \frac{|\alpha|k_F}{E_F}. \quad (C114)$$

Somewhat unexpectedly, the fully renormalized result (C114) coincides with the leading (first) term in the second-order result (3.51). The formally subleading but T -dependent $T/2E_F$ term in Eq. (3.51) does not show up in the fully renormalized result, which implies that, at best, it is of order $T/UL \propto T/\ln T$ for large but finite UL . Hence follows the result for $\delta\chi_{xx}$ presented in the main text, Eq. (4.39).

¹ See, e.g., recent reviews D. Belitz, T. R. Kirkpatrick, and T. Vojta, Rev. Mod. Phys. **77**, 579 (2005).

² H. v. Löhneysen, A. Rosch, M. Vojta, and P. Wölfle, Rev. Mod. Phys. **79**, 1015 (2007).

- ³ D. L. Maslov and A. V. Chubukov, and R. Saha, Phys. Rev. B **74**, 220402(R) (2006); D. L. Maslov and A. V. Chubukov, Phys. Rev. B **79**, 075112 (2009).
- ⁴ G. J. Conduit, A. G. Green, and B. D. Simons, Phys. Rev. Lett. **103**, 207201 (2009).
- ⁵ a) A. V. Chubukov, C. Pépin, and J. Rech, Phys. Rev. Lett. **92**, 147003 (2004); b) J. Rech, C. Pépin, and A. V. Chubukov, Phys. Rev. B **74**, 195126 (2006).
- ⁶ P. Simon and D. Loss, Phys. Rev. Lett. **98**, 156401 (2007).
- ⁷ P. Simon, B. Braunecker, and D. Loss, Phys. Rev. B **77**, 045108 (2008).
- ⁸ S. Chesi, R. A. Žak, P. Simon, and D. Loss, Phys. Rev. B **79**, 115445 (2009).
- ⁹ A. V. Chubukov, D. L. Maslov, S. Gangadharaiah, and L. I. Glazman, Phys. Rev. Lett. **95**, 026402 (2005);
- ¹⁰ A. V. Chubukov, D. L. Maslov, and A. J. Millis, Phys. Rev. B **73**, 045128 (2006).
- ¹¹ M. A. Baranov, M. Yu. Kagan, and M. S. Mar'enko, JETP Lett. **58**, 709 (1993).
- ¹² D. S. Hirashima and H. Takahashi, J. Phys. Soc. Jpn. **67**, 3816 (1998).
- ¹³ G. Y. Chitov and A. J. Millis, Phys. Rev. Lett. **86**, 5337 (2001).
- ¹⁴ G. Y. Chitov and A. J. Millis, Phys. Rev. B **64**, 054414 (2001).
- ¹⁵ A. V. Chubukov and D. L. Maslov, Phys. Rev. B **68**, 155113 (2003).
- ¹⁶ A. V. Chubukov and D. L. Maslov, Phys. Rev. B **69**, 121102(R) (2004).
- ¹⁷ J. Betouras, D. Efremov, and A. Chubukov, Phys. Rev. B **72**, 115112 (2005).
- ¹⁸ D. Efremov, J. Betouras, and A. Chubukov, Phys. Rev. B **77**, 220401(R) (2008).
- ¹⁹ I. L. Aleiner and K. B. Efetov, Phys. Rev. B **74**, 075102 (2006).
- ²⁰ A. Shekhter and A. M. Finkel'stein, Proc. Natl. Acad. Sci. U.S.A. **103**, 15765 (2006).
- ²¹ A. Shekhter and A. M. Finkel'stein, Phys. Rev. B **74**, 205122 (2006).
- ²² G. Schwiete and K. B. Efetov, Phys. Rev. B **74**, 165108 (2006).
- ²³ Yu. A. Bychkov and E. I. Rashba, JETP Lett. **39**, 78 (1984); Yu. A. Bychkov and E. I. Rashba, J. Phys. C: Solid State Phys. **17** (1984) 6039.
- ²⁴ R. A. Žak, D. L. Maslov, and D. L. Loss, unpublished.
- ²⁵ E. I. Rashba, J. of Supercond. **18**, 137 (2005).
- ²⁶ A. Ashrafi, private communication.
- ²⁷ E. I. Rashba, private communication.
- ²⁸ M. Pletyukhov and V. Gritsev, Phys. Rev. B **74**, 045307 (2006).
- ²⁹ A. V. Chubukov, D. L. Maslov, S. Gangadharaiah, and L. I. Glazman, Phys. Rev. B **71**, 205112 (2005).
- ³⁰ A. V. Chubukov and D. L. Maslov, Phys. Rev. B **76**, 165111 (2007).
- ³¹ L. P. Gor'kov and E. I. Rashba, Phys. Rev. Lett. **87**, 037004 (2001).
- ³² G.-H. Chen and M. E. Raikh, Phys. Rev. B **59**, 5090 (1999).
- ³³ M. Pletyukhov and S. Konschuh, Eur. Phys. J. **B** 60, 29 (2007).
- ³⁴ E. Mishchenko, unpublished.